

# On the Drell–Levy–Yan Relation to $O(\alpha_s^2)$

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## Abstract

We study the validity of a relation by Drell, Levy and Yan (DLY) connecting the deep inelastic structure (DIS) functions and the single-particle fragmentation functions in  $e^+e^-$  annihilation which are defined in the spacelike ( $q^2 < 0$ ) and timelike ( $q^2 > 0$ ) regions respectively. Here  $q$  denotes the momentum of the virtual photon exchanged in the deep inelastic scattering process or the annihilation process. An extension of the DLY-relation, which originally was only derived in the scaling parton model, to all orders in QCD leads to a connection between the two evolution kernels determining the  $q^2$ -dependence of the DIS structure functions and the fragmentation functions respectively. In relation to this we derive the transformation relations between the space- and time-like splitting functions up to next-to-leading order (NLO) and the coefficient functions up to NNLO both for unpolarized and polarized scattering. It is shown that the evolution kernels describing the combined singlet evolution for the structure functions  $F_2(x, Q^2)$ ,  $F_L(x, Q^2)$  where  $Q^2 = |q^2|$  or  $F_2(x, Q^2)$ ,  $\partial F_2(x, Q^2)/\partial \ln(Q^2)$  and the corresponding fragmentation functions satisfy the DLY relation up to next-to-leading order. We also comment on a relation proposed by Gribov and Lipatov.

# 1 Introduction

Neutral current deep inelastic lepton–nucleon scattering and single nucleon inclusive production in  $e^+e^-$  pair–annihilation are formally related by crossing the kinematic channels. Already before the advent of Quantum Chromodynamics (QCD) Drell, Levy, and Yan mentioned the possibility [1, 2] that the deep inelastic scattering structure functions at the one side and the nucleon fragmentation functions in  $e^+e^-$  pair–annihilation on the other side may be related by an analytic continuation from the  $t$ – to the  $s$ –channel. The hadronic tensors for the space–like process of deep inelastic scattering (DIS) and the time–like single nucleon inclusive reaction would therefore be related by

$$W_{\mu\nu}^{(S)}(q, p) = -W_{\mu\nu}^{(T)}(q, -p) . \quad (1)$$

Here  $p$  denotes the nucleon momentum and  $q$  is the 4–momentum transfer to the hadronic system, with  $q^2 < 0$  for deep inelastic scattering and  $q^2 > 0$  for  $e^+e^-$ –annihilation.

At that time the physical nucleons were considered as bound states built up of bare nucleons and pions in the context of the Yukawa theory. The interactions were described by Bethe–Salpeter [3] or Faddeev–type [4] equations and their generalization [5] <sup>1</sup> aiming at a perturbative description of the structure and fragmentation functions. In these theories neither infra–red nor collinear singularities are occurring. One may think off a general representation of the structure and fragmentation functions in terms of current–current expectation values. However it was already shown [1] that for  $e^+e^-$ –annihilation also diagrams of distinct connectedness appear (cf. also [8]) which are absent in DIS so that a proof of Eq. (1) at the non–perturbative level becomes very difficult. The relation could be established for the aforementioned ladder–models [1, 8, 9] <sup>2</sup>. Within QCD the picture changes. Here it turns out that a thorough perturbative description of the structure functions and fragmentation functions is not possible. However, perturbation theory may be used to describe the scaling violations of these functions at large values of  $|q^2|$  where the running coupling constant is small. The QCD–improved parton model, moreover, exhibits a similarity with the approach by Drell, Levy, and Yan, since in the range where single parton states are dominating, i.e. for the contributions of lowest twist, the non–perturbative contributions factorize. One therefore may calculate the respective one–particle evolution kernels and study their behavior under the crossing from the  $t$ – to the  $s$ –channel. A further complication in the case of a vector–theory as QCD is the emergence of infrared and also collinear singularities which have an essential impact on the crossing because of the behavior of the kernels at  $x = 1$  <sup>3</sup>. Here  $x$  denotes the Bjorken scaling variable which will be defined differently for the timelike and spacelike region except for  $x = 1$  where both definitions lead to the same value for  $x$ . As a consequence of the Bloch–Nordsiek theorem [13] the kernels become distribution–valued for  $x = 1$  [14].

In leading order for unpolarized scattering the crossing relations mentioned above were given in [15]. Similar relations hold in the polarized case. In this order these kernels are nothing but the lowest order splitting functions  $P_{kl}^{(0)}(x)$ , which are obtained from the inverse Mellin transforms of the anomalous dimensions  $\gamma_{kl}^{N(0)}$  [16].

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<sup>1</sup>Later on massive vector meson ladder–models were studied in [6, 7], where the crossing relation Eq. (1) was verified for the respective kernels.

<sup>2</sup>For a review on the early developments see [10].

<sup>3</sup>Eq. (1) was originally postulated assuming that those terms are absent [1]. See also the subsequent discussion in [11] pointing out that the exponent of the structure functions  $\propto (1-x)^p$  near  $x = 1$  [12] needs not to be integer.

It is the aim of the present paper to investigate the validity of the Drell–Levy–Yan (DLY) relation, if applied to perturbatively calculable partonic structure functions and quantities related to them up to the level of two-loop order. To establish this crossing relation between space- and time-like processes one has to study scheme-invariant quantities which are the physical evolution kernels for specific choices of observables as the unpolarized and polarized structure and fragmentation functions or derivatives of them w.r.t.  $q^2$ . Furthermore, conditions are derived for the transformation of the splitting and coefficient functions from the space- to the time-like case. For the coefficient functions we extend the discussion to the NNLO level. Other relations between the splitting functions such as supersymmetric relations and relations due to conformal symmetry were discussed elsewhere, cf. e.g. [17, 18].

The paper is organized as follows. Basic relations for the deep inelastic structure and fragmentation functions are summarized in section 2. In section 3 scheme-invariant combinations of coefficient and splitting functions are constructed for the space- and time-like processes both for unpolarized and polarized deep inelastic reactions where we consider two principal examples. The Drell–Levy–Yan relation is studied in detail in section 4. We also comment on a relation by Gribov and Lipatov [19] which emerged in the same context. Section 5 contains the conclusions. In the appendix we present the differences between the space- and time-like coefficient functions at  $O(\alpha_s^2)$  as well as the convolution relations which are needed for the investigation of the DLY-relation.

## 2 Structure Functions and Fragmentation Functions

Deep inelastic scattering (DIS) of a lepton ( $l$ ) off a hadron target ( $P$ ) is described by the process

$$l(k_1) + P(p) \rightarrow l(k_2) + 'X' \quad , \quad q = k_1 - k_2 \quad , \quad q^2 = -Q^2 < 0. \quad (2)$$

where ‘ $X$ ’ represents an inclusive final state. When a single gauge boson is exchanged between the incoming lepton and the hadron the above process factorizes into the leptonic part and the remaining hadronic part. In the case of forward scattering the scattering matrix element can be written in terms of the leptonic tensor  $L_{\mu\nu}$  and the hadronic tensor  $W_{\mu\nu}$  by

$$|M|^2 = L^{\mu\nu} W_{\mu\nu}. \quad (3)$$

The hadronic tensor [20] contains the unpolarized and polarized deep inelastic structure functions  $F_i$  and  $g_i$ . If the process is mediated by photon only we have  $i = 1, 2$  in both the polarized and unpolarized case. Notice that instead of  $F_1$  we can also take the longitudinal structure function  $F_L$ . At asymptotic values of the kinematic variables structure functions only depend on  $Q^2$  and the Bjorken scaling variable

$$x_B = \frac{Q^2}{2p \cdot q}, \quad 0 \leq x_B \leq 1. \quad (4)$$

In QCD the  $Q^2$  dependence of the structure functions is only logarithmic and it accounts for the violation of scaling. In the context of the parton model the structure functions can be expressed in terms of quark and gluon densities and the corresponding spacelike coefficient functions  $C_{i,k}^{(S)}$  ( $k = q, g$ )<sup>4</sup>

$$F_i^{(S)}(x_B, Q^2) = x_B \sum_{j=1}^{N_f} e_j^2 \int_{x_B}^1 \frac{dz}{z} \left[ \frac{1}{N_f} f_q^S \left( \frac{x_B}{z}, \mu_f^2 \right) C_{i,q}^{(S)S} \left( z, \frac{Q^2}{\mu_f^2} \right) + f_g \left( \frac{x_B}{z}, \mu_f^2 \right) \right]$$

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<sup>4</sup>Similar relations hold for the polarized structure functions  $g_1(x, Q^2)$  and  $g_2(x, Q^2)$  on the level of twist 2.

$$\times C_{i,g}^{(S)}\left(z, \frac{Q^2}{\mu_f^2}\right) + f_{q_j}^{NS}\left(\frac{x_B}{z}, \mu_f^2\right) C_{i,q}^{(S)NS}\left(z, \frac{Q^2}{\mu_f^2}\right)\Bigg],$$

$$i = 2, L, \quad (5)$$

where  $e_j$  denotes the charge of the  $j$ th quark flavor and  $N_f$  represents the number of light flavors. The scale  $\mu_f$ , appearing in the above equation, denotes the factorization scale which is introduced while removing the collinear singularities from the partonic structure functions. In addition one encounters a dependence on the renormalization scale  $\mu_r$  which arises in the renormalization procedure. For convenience this scale is put equal to the factorization scale in the following. Notice that the structure functions  $F_i$  and  $g_i$  do not depend on these scales. However the parton densities and the coefficient functions, which do depend on these scales, satisfy renormalization group equations which will be shown below.

In Eq. (5) the index  $(S)$  in the structure functions indicates the space-like nature of the process ( $q^2 < 0$ ). Furthermore in Eq. (5) appear the singlet ‘S’ and non-singlet ‘NS’ combinations of parton densities which are defined by

$$f_q^S(z, \mu_f^2) = \sum_{i=1}^{N_f} \left[ f_{q_i}(z, \mu_f^2) + f_{\bar{q}_i}(z, \mu_f^2) \right], \quad (6)$$

and

$$f_{q_i}^{NS}(z, \mu_f^2) = f_{q_i}(z, \mu_f^2) + f_{\bar{q}_i}(z, \mu_f^2) - \frac{1}{N_f} f_q^S(z, \mu_f^2), \quad (7)$$

respectively. Corresponding formulae hold for polarized scattering. In this case the polarized parton densities and polarized coefficient functions are denoted by  $\Delta f_k(z, \mu_f^2)$  ( $k = q, g$ ) and  $\Delta C_{i,k}(z, Q^2/\mu_f^2)$  ( $i = 1, 2$ ).

Whereas in deep inelastic scattering the constituent structure of the nucleons is studied, hadroproduction at  $e^+e^-$  colliders provides us with information about the fragmentation process of these constituents into the hadrons. This information is contained in the fragmentation functions observed in the reaction [1]

$$l(k_1) + \bar{l}(k_2) \rightarrow \bar{P}(p) + 'X' \quad , \quad q = k_1 + k_2 \quad , \quad q^2 \equiv Q^2 > 0, \quad (8)$$

where the symbols have the same meaning as in Eq. (2). These fragmentation functions are the analogues of the DIS structure functions. Therefore in the QCD improved parton model these functions can be expressed in a similar way in terms of parton fragmentation densities  $D_k$  ( $k = q, g$ ) multiplied by timelike coefficient functions i.e.

$$F_i^{(T)}(x_E, Q^2) = x_E \sum_{j=1}^{n_f} e_j^2 \int_{x_E}^1 \frac{dz}{z} \left[ \frac{1}{N_f} D_q^S\left(\frac{x_E}{z}, \mu_f^2\right) C_{i,q}^{(T)S}\left(z, \frac{Q^2}{\mu_f^2}\right) + D_g\left(\frac{x_E}{z}, \mu_f^2\right) \right. \\ \left. \times C_{i,g}^{(T)}\left(z, \frac{Q^2}{\mu_f^2}\right) + D_{q_j}^{NS}\left(\frac{x_E}{z}, \mu_f^2\right) C_{i,q}^{(T)NS}\left(z, \frac{Q^2}{\mu_f^2}\right) \right],$$

$$i = 2, L, \quad (9)$$

where the corresponding scaling variable for the process in Eq. (9) is defined by

$$x_E = \frac{2p \cdot q}{Q^2}, \quad 0 \leq x_E \leq 1, \quad (10)$$

The symbol  $T$  appearing within parentheses in Eq. (9) denotes that the fragmentation functions are measured in time-like processes. The scales  $\mu_f$  and  $\mu_r$  are defined in the same way as in Eq. (5) where like in DIS we set the renormalization scale equal to the factorization scale. Furthermore the definitions for the singlet and non-singlet parton fragmentation functions are the same as those for the parton densities given in Eqs. (6, 7). Similarly as in DIS one can also study the annihilation processes in Eq. (8) where the hadron  $P$  is polarized. This entails the definition of the polarized fragmentation functions denoted by  $g_1^{(T)}$  and  $g_2^{(T)}$  for which one can present a similar formula as in Eq. (9). Very often one also encounters the transverse structure function which in the timelike and spacelike case is given by

$$F_1^{(R)}(x, Q^2) = \frac{1}{2x} \left[ F_2^{(R)}(x, Q^2) - F_L^{(R)}(x, Q^2) \right],$$

with  $R = S \ (x = x_B) \quad R = T \ (x = x_E).$  (11)

### 3 Scheme-invariant Combinations

In this section we give a short outline of the origin of the factorization scheme dependence of the anomalous dimensions (splitting functions) and the coefficient functions. We also show how this dependence disappears in the evolution of the structure functions w.r.t. the kinematic variable  $Q^2$ . The discussion below deals with the DIS structure functions but the conclusions also hold for the fragmentation functions. The partonic structure functions denoted by  $\hat{\mathcal{F}}_{i,k}$  ( $i = 1, 2, L, k = q, g$ ), representing the QCD radiative corrections, contain various divergences. First these divergences have to be regularized for which the most convenient way is to choose the method of  $n$ -dimensional regularization. Using this method the singularities reveal themselves in the form of pole terms of the type  $1/\epsilon^j$ , with  $n = 4 + \epsilon$ , in the quantity  $\hat{\mathcal{F}}_{i,k}$ . The infrared divergences cancel between virtual and bremsstrahlung contributions by virtue of the Bloch-Nordsieck theorem [13]. Due to the Kinoshita-Lee-Nauenberg theorem [21] all the final state mass singularities are canceled too since the DIS structure function is an inclusive quantity. Then one is left with only two types of singularities. The first one originates from the ultraviolet region. This type of singularities is removed via a redefinition of the parameters appearing in the QCD Lagrangian. An example is the coupling constant which becomes equal to  $\alpha_s(\mu_r^2)$  where  $\mu_r$  is the renormalization scale. After coupling constant renormalization the hadronic structure function can be written as follows

$$F_i(x, Q^2) = \sum_{k=q,g} \left( \mathcal{F}_{ik} \left( \alpha_s(\mu_r^2), \frac{Q^2}{\mu^2}, \frac{\mu^2}{\mu_r^2}, \epsilon \right) \otimes \hat{f}_k \right) (x), \quad (12)$$

where the symbol  $\otimes$  denotes the Mellin-convolution defined by

$$(f \otimes g)(z) = \int_0^1 dz_1 \int_0^1 dz_2 f(z_1) g(z_2) \delta(z - z_1 z_2). \quad (13)$$

Furthermore  $\hat{f}_k$  is defined as the bare parton density which is scale independent and is an unphysical object because of the singular behavior of  $\mathcal{F}_{ik}$ . Notice that the latter depends on the scale  $\mu_r$  and therefore on the renormalization scheme w.r.t. the coupling constant. The parameter  $\mu$  originates from  $n$ -dimensional regularization because in this method the coupling constant gets a dimension. The second type of singularity originates from the collinear region

which can be attributed to the vanishing mass of the initial state parton represented by either the (anti-) quark or the gluon. Hence the  $\epsilon$  in Eq. (12) represents the collinear singularities which are removed from the partonic structure function via mass factorization and transferred to a transition function  $\Gamma_{lk}$  as follows

$$\hat{\mathcal{F}}_{ik}(z, \alpha_s(\mu_r^2), \frac{Q^2}{\mu^2}, \frac{\mu^2}{\mu_r^2}, \epsilon) = \sum_{l=q,g} \left( C_{i,l}(\alpha_s(\mu_r^2), \frac{Q^2}{\mu^2}, \frac{\mu_f^2}{\mu_r^2}) \otimes \Gamma_{lk}(\alpha_s(\mu_r^2), \frac{\mu_f^2}{\mu^2}, \frac{\mu_f^2}{\mu_r^2}, \epsilon) \right) (z). \quad (14)$$

This procedure provides us with the coefficient function denoted by  $C_{i,l}$ . Substitution of Eq. (14) into Eq. (12) leads to the result

$$F_i(x, Q^2) = \sum_{l=q,g} \left( C_{i,l}(\alpha_s(\mu_r^2), \frac{Q^2}{\mu_f^2}, \frac{\mu_f^2}{\mu_r^2}) \otimes f_l(\alpha_s(\mu_r^2), \frac{\mu_f^2}{\mu^2}, \frac{\mu_f^2}{\mu_r^2}) \right) (x), \quad (15)$$

where the renormalized parton density is defined as

$$f_l(z, \alpha_s(\mu_r^2), \frac{\mu_f^2}{\mu^2}, \frac{\mu_f^2}{\mu_r^2}) = \sum_{k=q,g} \left( \Gamma_{lk}(\alpha_s(\mu_r^2), \frac{\mu_f^2}{\mu^2}, \frac{\mu_f^2}{\mu_r^2}, \epsilon) \otimes \hat{f}_k \right) (z). \quad (16)$$

Since the mass factorization can be carried out in various ways one is left with an additional scheme dependence which comes on top of the renormalization scheme dependence entering the coupling constant in Eq. (12). The former only shows up in the parton densities and the coefficient functions and it only disappears in specific combinations representing physical quantities. Hence physical quantities are invariant under scheme transformation. Like in the case of renormalization, mass factorization leads to the introduction of a scale  $\mu_f$  called mass factorization scale which is related to the factorization scheme dependence. Like in the latter case  $\mu_f$  drops out in physical quantities as the DIS structure functions or fragmentation functions. The change of the parton densities and the coefficient functions with respect to a variation in the scales  $\mu_r$  and  $\mu_f$  is determined by the renormalization group equation (RGE) [22] The renormalization group equation of the parton densities follow from the one presented for the transition functions  $\Gamma_{lk}$ . The latter takes the following form

$$\left( \left[ \left\{ \mu_f^2 \frac{\partial}{\partial \mu_f^2} + \beta(a_s(\mu_f^2)) \frac{\partial}{\partial a_s(\mu_f^2)} \right\} \mathbf{1}_{\delta_{lm}} - \frac{1}{2} P_{lm}(a_s(\mu_f^2), \epsilon) \right] \otimes \Gamma_{mk}(a_s(\mu_f^2), \frac{\mu_f^2}{\mu^2}, 1, \epsilon) \right) (z) = 0, \\ a_s(\mu_f^2) \equiv \frac{\alpha_s(\mu_f^2)}{4\pi}, \quad \mathbf{1} = \delta(1-z), \quad (17)$$

where we have set  $\mu_r = \mu_f$  for simplicity. The functions  $P_{ij}(a_s, \epsilon, z)$  appearing in the above equation are the splitting functions. Furthermore the beta-function is defined by

$$\mu_r^2 \frac{d a_s(\mu_r^2)}{d \mu_r^2} = -\beta_0 a_s^2(\mu_r^2) - \beta_1 a_s^3(\mu_r^2) \cdots, \quad (18)$$

The same equation as in Eq. (17) also applies to the parton density because of the definition in Eq. (16). The scale dependence of the coefficient function in Eq. (15) is given by

$$\left( \left[ \left\{ \mu_f^2 \frac{\partial}{\partial \mu_f^2} + \beta(a_s(\mu_f^2)) \frac{\partial}{\partial a_s(\mu_f^2)} \right\} \mathbf{1}_{\delta_{lm}} + \frac{1}{2} P_{lm}(a_s(\mu_f^2), \epsilon) \right] \otimes C_{i,m}(a_s(\mu_f^2), \frac{Q^2}{\mu_f^2}, 1) \right) (z) = 0. \quad (19)$$

As has been mentioned above scheme transformations such as

$$\Gamma_{lk} \rightarrow \sum_{m=q,g} Z_{lm} \otimes \bar{\Gamma}_{mk} \quad , \quad C_{i,l} \rightarrow \sum_{m=q,g} \bar{C}_{i,m} \otimes Z_{ml}^{-1} , \quad (20)$$

will not alter the physical observable like e.g. the structure functions and fragmentation functions. The relation between the splitting and coefficient functions computed in two different schemes is found to be

$$P_{lk} = \sum_{\{m,n\}=q,g} Z_{lm} \otimes \bar{P}_{mn} \otimes (Z^{-1})_{nk} - 2\beta(a_s) \sum_{m=q,g} Z_{lm} \otimes \frac{d}{da_s} (Z^{-1})_{mk} , \quad (21)$$

$$C_{i,l} = \sum_{m=q,g} \bar{C}_{i,m} \otimes (Z^{-1})_{ml} . \quad (22)$$

Below we present the relation between the coefficient functions computed in two different schemes up to order  $a_s^2(Q^2)$ . Notice that we have chosen here  $\mu_f^2 = Q^2$  in order to get rid off the logarithms  $\ln(Q^2/\mu_f^2)$  which usually appear. Up to  $O(a_s^2)$  one obtains

$$\begin{aligned} C_{i,q} = & \delta(1-z) + a_s \left( \bar{C}_{i,q}^{(1)} + Z_{qq}^{(1)} \right) + a_s^2 \left( \bar{C}_{i,q}^{(2)} + Z_{qq}^{(2)} + (Z_{qq}^{(1)})^2 + Z_{qq}^{(1)} \otimes Z_{gq}^{(1)} \right. \\ & \left. + \bar{C}_{i,q}^{(1)} \otimes Z_{qq}^{(1)} + \bar{C}_{i,g}^{(1)} \otimes Z_{gq}^{(1)} \right) + \dots , \end{aligned} \quad (23)$$

$$\begin{aligned} C_{i,g} = & a_s \left( \bar{C}_{i,g}^{(1)} + Z_{gg}^{(1)} \right) + a_s^2 \left( \bar{C}_{i,g}^{(2)} + Z_{gg}^{(2)} + Z_{gg}^{(1)} \otimes (Z_{gg}^{(1)} + Z_{qq}^{(1)}) \right. \\ & \left. + \bar{C}_{i,q}^{(1)} \otimes Z_{gg}^{(1)} + \bar{C}_{i,g}^{(1)} \otimes Z_{gg}^{(1)} \right) + \dots . \end{aligned} \quad (24)$$

In the subsequent part of the paper it is much more convenient to derive the expressions in the Mellin transform representation so that one can avoid the convolution symbol  $\otimes$ . The Mellin transform of a function  $f(z)$  is given by

$$f^{(N)} = \int_0^1 dz \, z^{N-1} f(z) \quad (25)$$

In this way Eq. (13) can be written as

$$(f \otimes g)^N = \int_0^1 dz \, z^{N-1} (f \otimes g)(z) = f^N \cdot g^N . \quad (26)$$

Since the structure functions are scheme independent they become renormalization group invariants. Hence they satisfy the RG equation

$$\left[ \mu_f^2 \frac{\partial}{\partial \mu_f^2} + \beta(a_s(\mu_f^2)) \frac{\partial}{\partial a_s(\mu_f^2)} \right] F_i^N(x, Q^2) = 0 \quad (27)$$

The equation above follows from combining Eqs. (17, 19) and (15). However the independence of the structure function on the scales  $\mu_f$  and  $\mu_r$  is not manifest when multiplying parton densities with coefficient functions. In particular when the perturbation series of a physical quantity is computed up to finite order there is a residual dependence on these unphysical scales (see e.g.

[23]. Their influence is expected to become smaller when higher order terms in the perturbation series are taken into account [24]. To avoid the problem of the factorization scheme dependence of the structure function when the perturbation series is truncated up to fixed order it is better to study evolution equations for the structure functions with respect to a *physical* scale which is represented by a kinematic variable like  $Q^2$ . In these type of evolution equations the kernels are factorization scheme independent order by order in perturbation theory. However the dependence on the choice of renormalization scheme and therefore the dependence on  $\mu_r$  remains so that one is able to obtain a better estimate of the theoretical error on  $\alpha_s$ . For such an equation one needs two different structure functions called  $F_A(x, Q^2)$  and  $F_B(x, Q^2)$ . Examples are  $A = 2$  and  $B = L$  or  $F_A$  and  $Q^2 d F_A / d Q^2$ . Limiting ourselves to the singlet case, the evolution equation for the non-singlet structure functions is even more simple, one can write

$$F_I^N(Q^2) = f_q^N \left( a_s(\mu_f^2), \frac{\mu_f^2}{Q_0^2} \right) C_{I,q}^N \left( a_s(\mu_f^2), \frac{Q^2}{\mu_f^2} \right) + f_g^N \left( a_s(\mu_f^2), \frac{\mu_f^2}{Q_0^2} \right) C_{I,g}^N \left( a_s(\mu_f^2), \frac{Q^2}{\mu_f^2} \right),$$

$$I = A, B, \quad (28)$$

Here one can view the  $C_{I,l}^N$  ( $I = A, B$ ,  $l = q, g$ ) as matrix elements so that the equation above has the form

$$\begin{pmatrix} F_A^N \\ F_B^N \end{pmatrix} = \begin{pmatrix} C_{Aq}^N & C_{Ag}^N \\ C_{Bq}^N & C_{Bg}^N \end{pmatrix} \begin{pmatrix} f_q^N \\ f_g^N \end{pmatrix}. \quad (29)$$

The coefficient functions satisfy the RG-equation in Eq. (19) and the solution is given by the T-ordered exponential [25]

$$C_{I,l}^N \left( a_s(\mu_f^2), \frac{Q^2}{\mu_f^2} \right) = C_{I,m}^N \left( a_s(Q^2), 1 \right) \left( T_{a_s} \left[ \exp \left\{ - \int_{a_s(\mu_f^2)}^{a_s(Q^2)} dx \frac{\gamma^N(x)}{2\beta(x)} \right\} \right] \right)_{ml}, \quad (30)$$

where  $\gamma^N$  is the anomalous dimension matrix defined by

$$\gamma_{lk}^N = - \int_0^1 dz z^{N-1} P_{lk}(z). \quad (31)$$

We will now differentiate the coefficient functions w.r.t.  $Q^2$

$$Q^2 \frac{\partial C_{I,k}^N(a_s(\mu_f^2), Q^2/\mu_f^2)}{\partial Q^2} = \beta(a_s(Q^2)) \frac{\partial C_{I,k}^N(a_s(\mu_f^2), Q^2/\mu_f^2)}{\partial a_s(Q^2)} =$$

$$\left[ \beta(a_s(Q^2)) \frac{\partial C_{I,m}^N(a_s(Q^2), 1)}{\partial a_s(Q^2)} (C^N)^{-1}_{m,J}(a_s(Q^2), 1) \right.$$

$$\left. - \frac{1}{2} C_{I,m}^N(a_s(Q^2), 1) \gamma_{mn}^N(a_s(Q^2)) (C^N)^{-1}_{n,J}(a_s(Q^2), 1) \right] C_{J,k}^N(a_s(\mu_f^2), \frac{Q^2}{\mu_f^2}). \quad (32)$$

One can show that the expression above is invariant under scheme transformations. The latter are given by

$$\gamma_{lk}^N = \sum_{\{m,n\}=q,g} Z_{lm}^N \bar{\gamma}_{mn}^N (Z^N)^{-1}_{nk} + 2\beta(a_s) \sum_{m=q,g} Z_{lm}^N \frac{\partial}{\partial a_s} (Z^N)^{-1}_{mk}, \quad (33)$$

$$C_{I,l}^N = \sum_{m=q,g} \bar{C}_{I,m}^N (Z^N)^{-1}_{ml}, \quad (C^N)^{-1}_{l,I} = \sum_{m=q,g} Z_{lm}^N (\bar{C}^N)^{-1}_{m,I}. \quad (34)$$



Since the  $Q^2$ -dependence only resides in the coefficient function the same evolution equation as in Eq. (32) also applies to  $F_I^N$  in Eq. (28). For a short-hand notation we introduce the evolution variable  $t$

$$t = -\frac{2}{\beta_0} \ln \left( \frac{a_s(Q^2)}{a_s(Q_0^2)} \right), \quad (35)$$

so that we obtain

$$\frac{\partial}{\partial t} \begin{pmatrix} F_A^N \\ F_B^N \end{pmatrix} = -\frac{1}{4} \begin{pmatrix} K_{AA}^N & K_{AB}^N \\ K_{BA}^N & K_{BB}^N \end{pmatrix} \begin{pmatrix} F_A^N \\ F_B^N \end{pmatrix}, \quad (36)$$

where the physical (scheme invariant) kernel is given by

$$K_{IJ}^N = \left[ -4 \frac{\partial C_{I,m}^N(t)}{\partial t} (C^N)_{m,J}^{-1}(t) - \frac{\beta_0 a_s(Q^2)}{\beta(a_s(Q^2))} C_{I,m}^N(t) \gamma_{mn}^N(t) (C^N)_{n,J}^{-1}(t) \right]. \quad (37)$$

The kernels  $K_{IJ}^N$  depend both on the anomalous dimensions  $\gamma_{lk}^N$  and the coefficient functions  $C_{I,l}^N(Q^2)$  but the latter two quantities are combined in a factorization scheme independent way. This factorization scheme independence of  $K_{IJ}^N$  holds order by order in perturbation theory. Using the series expansions for the anomalous dimensions and coefficient functions in terms of the strong coupling constant

$$\gamma_{lk}^N = \sum_{n=0}^{\infty} a_s^{n+1}(Q^2) (\gamma^N)_{lk}^{(n)}, \quad C_{I,l}^N(Q^2) = \sum_{n=0}^{\infty} a_s^n(Q^2) (C^N)_{I,l}^{(n)}, \quad l, k = q, g, \quad I = A, B, \quad (38)$$

one can compute order by order the coefficients in the perturbation series of the kernel

$$K_{IJ}^N = \sum_{n=0}^{\infty} a_s^n(Q^2) (K^N)_{IJ}^{(n)}. \quad (39)$$

Notice that the coefficients  $(K^N)_{IJ}^{(n)}$  are not invariant with respect to a finite renormalization of the coupling constant. This dependence is removed when the perturbation series in Eq. (39) is resummed in all orders.

### 3.1 $F_2(x, Q^2)$ and $F_L(x, Q^2)$

Let us consider now two specific examples, choosing the structure functions  $F_2(x, Q^2)$  and  $F_L(x, Q^2)$  or the structure function  $F_2(x, Q^2)$  and its slope  $\partial F_2(x, Q^2)/\partial t$  as the observables  $F_{A,B}(x, Q^2)$ . in this combination of observables it is convenient to normalize the structure function  $F_L(x, Q^2)$  to its gluonic contribution in lowest order. This is because  $F_L(x, Q^2)$  vanishes in zeroth order of  $\alpha_s$  due to the Callan–Gross relation, cf. Eq. (11), contrary to the structure function  $F_2(x, Q^2)$ . Therefore this normalization accounts for keeping the same order in the coupling constant for the two quantities

$$F_A^N(Q^2) = F_2^{N(S)}(Q^2), \quad F_B^N(Q^2) = \frac{F_L^N(Q^2)}{a_s(Q^2) C_{L,g}^{N(1)}}. \quad (40)$$

Since both the coefficient functions  $C_{Lq}^{(1)}$  and  $C_{Lg}^{(1)}$  are scheme invariants such a normalization is possible. We expand now the kernels  $K_{IJ}^N$  for this choice of observables into a series in  $a_s$ . The

lowest order contribution is well-known, cf. e.g. [29],

$$\begin{aligned}
K_{22}^{N(0)} &= \gamma_{qq}^{N(0)} - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \gamma_{qg}^{N(0)} & K_{2L}^{N(0)} &= \gamma_{qg}^{N(0)} \\
K_{L2}^{N(0)} &= \gamma_{gg}^{N(0)} - \left( \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \right)^2 \gamma_{qg}^{N(0)} & K_{LL}^{N(0)} &= \gamma_{gg}^{N(0)} + \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \gamma_{qg}^{N(0)} \\
&\quad + \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} (\gamma_{qq}^{N(0)} - \gamma_{gg}^{N(0)}) .
\end{aligned} \tag{41}$$

To next-to-leading order in  $a_s(Q^2)$ , one finds

$$\begin{aligned}
K_{22}^{N(1)} &= \gamma_{qq}^{N(1)} - \frac{\beta_1}{\beta_0} \gamma_{qq}^{N(0)} - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \left( \gamma_{qg}^{N(1)} - \frac{\beta_1}{\beta_0} \gamma_{qg}^{N(0)} \right) + \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} C_{2,g}^{N(1)} \gamma_{qq}^{N(0)} \\
&\quad - \left[ \frac{C_{L,q}^{N(2)}}{C_{L,g}^{N(1)}} + \left( \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \right)^2 C_{2,g}^{N(1)} - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \frac{C_{L,g}^{N(2)}}{C_{L,g}^{N(1)}} \right] \gamma_{qg}^{N(0)} + C_{2,g}^{N(1)} \gamma_{gg}^{N(0)} \\
&\quad - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} C_{2,g}^{N(1)} \gamma_{gg}^{N(0)} + 2\beta_0 \left( C_{2,q}^{N(1)} - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} C_{2,g}^{N(1)} \right)
\end{aligned} \tag{42}$$

$$\begin{aligned}
K_{2L}^{N(1)} &= \gamma_{qg}^{N(1)} - \frac{\beta_1}{\beta_0} \gamma_{qg}^{N(0)} - C_{2,g}^{N(1)} (\gamma_{qq}^{N(0)} - \gamma_{gg}^{N(0)}) + 2\beta_0 C_{2,g}^{N(1)} \\
&\quad + \left( C_{2,q}^{N(1)} + \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} C_{2,g}^{N(1)} - \frac{C_{L,g}^{N(2)}}{C_{L,g}^{N(1)}} \right) \gamma_{qg}^{N(0)}
\end{aligned} \tag{43}$$

$$\begin{aligned}
K_{L2}^{N(1)} &= \gamma_{gg}^{N(1)} - \frac{\beta_1}{\beta_0} \gamma_{gg}^{N(0)} + \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \left( \gamma_{qq}^{N(1)} - \frac{\beta_1}{\beta_0} \gamma_{qq}^{N(0)} \right) \\
&\quad - \left( \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \right)^2 \left( \gamma_{qg}^{N(1)} - \frac{\beta_1}{\beta_0} \gamma_{qg}^{N(0)} \right) - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \left( \gamma_{gg}^{N(1)} - \frac{\beta_1}{\beta_0} \gamma_{gg}^{N(0)} \right) \\
&\quad + \left[ \frac{C_{L,q}^{N(2)}}{C_{L,g}^{N(1)}} - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} C_{2,q}^{N(1)} + \left( \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \right)^2 C_{2,g}^{N(1)} \right] \gamma_{qq}^{N(0)} \\
&\quad - \left[ \left( \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \right)^3 C_{2,g}^{N(1)} + 2 \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \frac{C_{L,q}^{N(2)}}{C_{L,g}^{N(1)}} - \left( \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \right)^2 \frac{C_{L,g}^{N(2)}}{C_{L,g}^{N(1)}} \right. \\
&\quad \left. - \left( \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \right)^2 C_{2,q}^{N(1)} \right] \gamma_{qg}^{N(0)} + \left( \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} C_{2,g}^{N(1)} - C_{2,q}^{N(1)} + \frac{C_{L,g}^{N(2)}}{C_{L,g}^{N(1)}} \right) \gamma_{gg}^{N(0)}
\end{aligned}$$

$$\begin{aligned}
& - \left[ \frac{C_{L,q}^{N(2)}}{C_{L,g}^{N(1)}} + \left( \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \right)^2 C_{2,g}^{N(1)} - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} C_{2,q}^{N(1)} \right] \gamma_{gg}^{N(0)} \\
& + 2\beta_0 \left( \frac{C_{L,q}^{N(2)}}{C_{L,g}^{N(1)}} - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \frac{C_{L,g}^{N(2)}}{C_{L,g}^{N(1)}} \right)
\end{aligned} \tag{44}$$

$$\begin{aligned}
K_{LL}^{N(1)} &= \gamma_{gg}^{N(1)} - \frac{\beta_1}{\beta_0} \gamma_{gg}^{N(0)} + \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \left( \gamma_{qq}^{N(1)} - \frac{\beta_1}{\beta_0} \gamma_{qq}^{N(0)} \right) \\
& - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} C_{2,g}^{N(1)} \gamma_{qq}^{N(0)} + \left[ \frac{C_{L,q}^{N(2)}}{C_{L,g}^{N(1)}} - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \frac{C_{L,g}^{N(2)}}{C_{L,g}^{N(1)}} + \left( \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \right)^2 C_{2,g}^{N(1)} \right] \gamma_{gg}^{N(0)} \\
& - C_{2,g}^{N(1)} \gamma_{gg}^{N(0)} + \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} C_{2,g}^{N(1)} \gamma_{gg}^{N(0)} + 2\beta_0 \frac{C_{L,g}^{N(2)}}{C_{L,g}^{N(1)}}
\end{aligned} \tag{45}$$

It is evident that the representation in terms of Mellin-moments is of advantage when compared to the corresponding  $x$ -space expressions. In the latter case one has to find the inverse Mellin transforms of quantities where  $C_{i,k}^N$  and  $\gamma_{kl}^N$  appear in the denominators of the expressions above which in general is not possible.

### 3.2 $F_2(x, Q^2)$ and $\partial F_2(x, Q^2)/\partial t$

A second example concerns the structure function  $F_2(x, Q^2)$  and its slope. Both quantities are well measurable in the present-day deep inelastic scattering experiments. The observables  $F_{A,B}(x, Q^2)$  are here

$$F_A^N(Q^2) = F_2^{(S)N}(Q^2), \quad F_B^N(Q^2) = \frac{\partial}{\partial t} F_2^{(S)N}(Q^2) \tag{46}$$

This example has been considered before in [27]. In leading order one obtains

$$\begin{aligned}
K_{22}^{N(0)} &= 0 & K_{2d}^{N(0)} &= -4 \\
K_{d2}^{N(0)} &= \frac{1}{4} \left( \gamma_{qq}^{N(0)} \gamma_{gg}^{N(0)} - \gamma_{gg}^{N(0)} \gamma_{qq}^{N(0)} \right) & K_{dd}^{N(0)} &= \gamma_{qq}^{N(0)} + \gamma_{gg}^{N(0)}.
\end{aligned} \tag{47}$$

The next-to-leading order kernels read :

$$K_{22}^{N(1)} = 0 \tag{48}$$

$$K_{2d}^{N(1)} = 0 \tag{49}$$

$$\begin{aligned}
K_{d2}^{N(1)} &= \frac{1}{4} \left[ \gamma_{gg}^{N(0)} \gamma_{qq}^{N(1)} + \gamma_{gg}^{N(1)} \gamma_{qq}^{N(0)} - \gamma_{qq}^{N(1)} \gamma_{gg}^{N(0)} - \gamma_{qq}^{N(0)} \gamma_{gg}^{N(1)} \right] \\
& - \frac{\beta_1}{2\beta_0} \left( \gamma_{qq}^{N(0)} \gamma_{gg}^{N(0)} - \gamma_{gg}^{N(0)} \gamma_{qq}^{N(0)} \right) + \frac{\beta_0}{2} C_{2,q}^{N(1)} \left( \gamma_{qq}^{N(0)} + \gamma_{gg}^{N(0)} - 2\beta_0 \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{\beta_0}{2} \frac{C_{2,g}^{N(1)}}{\gamma_{qg}^{N(0)}} \left[ (\gamma_{qq}^{N(0)})^2 - \gamma_{qq}^{N(0)} \gamma_{gg}^{N(0)} + 2\gamma_{qg}^{N(0)} \gamma_{gq}^{N(0)} - 2\beta_0 \gamma_{qq}^{N(0)} \right] \\
& -\frac{\beta_0}{2} \left( \gamma_{qq}^{N(1)} - \frac{\gamma_{qq}^{N(0)} \gamma_{qg}^{N(1)}}{\gamma_{qg}^{N(0)}} \right)
\end{aligned} \tag{50}$$

$$\begin{aligned}
K_{dd}^{N(1)} &= \gamma_{qq}^{N(1)} + \gamma_{gg}^{N(1)} - \frac{\beta_1}{\beta_0} \left( \gamma_{qq}^{N(0)} + \gamma_{gg}^{N(0)} \right) \\
& - \frac{2\beta_0}{\gamma_{qg}^{N(0)}} \left[ C_{2,g}^{N(1)} \left( \gamma_{qq}^{N(0)} - \gamma_{gg}^{N(0)} - 2\beta_0 \right) - \gamma_{qg}^{N(1)} \right] + 4\beta_0 C_{2,q}^{N(1)} - 2\beta_1 .
\end{aligned} \tag{51}$$

For this combination in next-to-leading order the evolution depends on two evolution kernels only. In the case of polarized deep inelastic scattering similar relations apply considering the structure function  $g_1(x, Q^2)$  and its slope. The anomalous dimensions and coefficient functions of the unpolarized case have to be substituted by those for polarized scattering.

Although enforced by Eq. (36) one has still to show that the kernels (41-45, 47-51) are scheme-independent by an explicit calculation, which we have done using Eqs. (33,34) for the next-to-leading order contributions. In leading order the scheme-invariance is visible explicitly, since the leading order anomalous dimensions and the lowest order coefficient functions for  $F_L(x, Q^2)$  are scheme invariants.

With the help of the evolution equation (36) we are now prepared to ask for the validity of crossing relations between different space- and time-like quantities in perturbative QCD. Relations of this kind are henceforth called Drell-Levy-Yan (DLY) relations, although the original reasoning of these and other authors was quite different. One condition to ask such a question at all is that the behavior of all contributing parts under crossing from space- to time-like momentum transfer are controlled. At large momentum transfers  $|q^2|$  the single parton picture applies and the non-perturbative parton densities factorize. This makes it possible to study the respective evolutions kernels without reference to the **non-perturbative** input densities. Even if a crossing relation for these quantities does not exist, one still may investigate whether it exists for the **perturbative** evolution kernels. A further condition for this investigation is that the latter quantities are **scheme-invariant**, as in Eq. (36).

## 4 Drell-Levy-Yan relations

In the following we study in detail an interesting relation between deep inelastic lepton hadron scattering and  $e^+e^-$  annihilation into a hadron and anything else, proposed by Drell, Levy and Yan [1]. Here we first briefly illustrate the idea behind the work of DLY for completeness. In field theory, the deep inelastic  $e^+e^-$  annihilation can be related to matrix elements of hadronic electromagnetic current operators similar to that of deep inelastic lepton-hadron scattering. The crucial difference, apart from the ones which originate from the kinematics, is that the annihilation process is *not* related to the forward Compton amplitude contrary to deep inelastic scattering because in the former process the hadron is observed in the final state. Nevertheless, both processes are related by crossing symmetry which any field theory enjoys. This motivated DLY to study the process in detail and then relate it to the deep inelastic scattering process.

From the structure of the hadronic tensors  $W_{\mu\nu}^S(q, p)$  (space-like) and  $W_{\mu\nu}^T(q, p)$  (time-like) and using the standard reduction formalism one can infer that

$$W_{\mu\nu}^T(q, p) = -W_{\mu\nu}^S(q, -p) , \quad (52)$$

where the momenta within the respective parentheses of the above quantities are the same as those defined in the beginning of the paper. In the Bjorken limit for both deep inelastic scattering and deep inelastic annihilation for  $q^2 = -Q^2, p \cdot q \rightarrow \infty$  and  $q^2 = Q^2, p \cdot q \rightarrow \infty$ , respectively the scaling structure and fragmentation functions satisfy the following relation <sup>5</sup>:

$$F_i^{(S)}(x_B) = -(-1)^{2(s_1+s_2)} x_E F_i^{(T)}\left(\frac{1}{x_E}\right) , \quad i = 1, 2, L . \quad (53)$$

Here it has been assumed that non-perturbative input parton densities can be decoupled trivially and are the same. In other words, the functions  $F_i^{(T)}(x_E)$  are the analytic continuations of the corresponding functions  $F_i^{(S)}(x_B)$  from  $0 < x_B \leq 1$  to  $1 \leq x_E < \infty$ . This is true only when the continuation is smooth, i.e. if there are no singularities for example at  $x = 1$  etc. This relation is called DLY-relation in the literature.

In this section, we study this property in more detail extending earlier work [17]. It is particularly interesting to study the above transformation at the level of the splitting functions and coefficient functions which constitute the physical quantities such as the structure and fragmentation functions. Then we show how these relations are preserved for the physical quantities by looking at the kernels discussed in the previous section. Apart from scaling violation one also encounters distributions of the type

$$\delta(1-z) \quad , \quad \left(\frac{\ln^i(1-z)}{1-z}\right)_+ , \quad i = 0, 1, 2, \dots , \quad (54)$$

which destroy the continuation through  $z = 1$ . Here the distribution  $(\ln^i(1-z)/(1-z))_+$  is represented by

$$\left(\frac{\ln^i(1-z)}{1-z}\right)_+ = \delta(1-z) \frac{\ln^{i+1} \delta}{(i+1)} + \theta(1-\delta-z) \frac{\ln^i(1-z)}{(1-z)} , \quad (55)$$

where  $\delta \ll 1$ . It turns out that the DLY-relation is violated for the coefficient functions and splitting functions separately because both are scheme dependent. This in particular happens when we adopt the  $\overline{\text{MS}}$ -scheme. Here the relation is already violated up to one-loop order for the coefficient functions. Although one can choose other schemes in which Eq. (53) is preserved (see [17]) up to one-loop order we do not know whether this will hold up to any arbitrary order in perturbation theory.

Let us start with the simplest examples and consider the scheme-invariant evolution kernels Eq. (41, 47).

## 4.1 The Drell-Levy-Yan Relations at Leading Order

In the case of the scheme-independent evolution kernels describing the evolution of  $F_2(x, Q^2)$  and  $\partial F_2(x, Q^2)/\partial t$ , respectively, or its polarized counterpart for the structure function  $g_1(x, Q^2)$ ,

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<sup>5</sup>Here we indicate the overall signs in case of the scattering of particles of different spin, cf. [30]. In the original work of DLY [1] the Yukawa-theory was discussed which does not contain gauge bosons.

only the transformation of two combinations of the leading order anomalous dimensions has to be considered, cf. (47). These are the determinant and the trace of the singlet anomalous dimension matrix at leading order. In both quantities the color factors of the off-diagonal elements enter only as a product. The unpolarized and polarized leading order splitting functions read

$$P_{qq}^{(0)}(z) = \Delta P_{qq}^{(0)}(z) = 4C_F \left[ \frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \right] \quad (56)$$

$$P_{qg}^{(0)}(z) = 8T_R N_f \left[ z^2 + (1-z)^2 \right] \quad (57)$$

$$\Delta P_{qg}^{(0)}(z) = 8T_R N_f \left[ z^2 - (1-z)^2 \right] \quad (58)$$

$$P_{gq}^{(0)}(z) = 4C_F \frac{1+(1-z)^2}{z} \quad (59)$$

$$\Delta P_{gq}^{(0)}(z) = 4C_F \frac{1-(1-z)^2}{z} \quad (60)$$

$$P_{gg}^{(0)}(z) = 8C_A \left[ \frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) \right] + 2\beta_0\delta(1-z) \quad (61)$$

$$\Delta P_{gg}^{(0)}(z) = 8C_A \left[ \frac{1}{(1-z)_+} + 1 - 2z \right] + 2\beta_0\delta(1-z) . \quad (62)$$

The crossing relations of the leading order splitting functions are

$$\begin{aligned} \bar{P}_{qq}^{(0)} &= -zP_{qq}^{(0)} \left( \frac{1}{z} \right) & \bar{P}_{qg}^{(0)} &= \frac{C_F}{2N_f T_f} zP_{qg}^{(0)} \left( \frac{1}{z} \right) \\ \bar{P}_{gq}^{(0)} &= \frac{2N_f T_f}{C_F} zP_{gq}^{(0)} \left( \frac{1}{z} \right) & \bar{P}_{gg}^{(0)} &= -zP_{gg}^{(0)} \left( \frac{1}{z} \right) , \end{aligned} \quad (63)$$

where one demands

$$\delta(1-z) \rightarrow -\delta(1-z) . \quad (64)$$

Eq. (63) is easily verified and implies the validity of the crossing relation from space- to time-like evolution kernels Eq. (47), i.e. the validity of the DLY-relation for this case. For the second set of physical evolution kernels the DLY-relation follows at leading order referring to the transformation relations for the leading order longitudinal coefficient functions, Eqs. (78, 79) in an analogous way.

## 4.2 NLO Splitting function

As we know, the splitting functions and coefficient functions are not physical quantities due to their factorization scheme dependence. Hence, the naive continuation rule for these quantities may be violated, which is indeed the case in most of the schemes, e.g. in the  $\overline{\text{MS}}$  scheme characteristic of  $n$ -dimensional regularization. It was demonstrated by Curci, Furmanski and Petronzio [33] that by an appropriate modification of the continuation rule in the  $\overline{\text{MS}}$  scheme

one can show that the time-like splitting functions are related to their space-like counter parts. Since the modification of the continuation rule has to do with the scheme one adopts, it simply amounts to finding finite renormalization factors. It was shown that the finite renormalization factors can be constructed from the  $\epsilon$ -dependent part of the splitting function when computed in dimensional regularization [34]. In addition to this, care should be taken when dealing with quark and gluon states which was not the case in the work by DLY, where a color and flavor neutral field theory was discussed. The transformation rules are :

- The diagonal elements of the space-like flavor singlet splitting functions  $P_{qq}, P_{gg}$  have to be multiplied by  $(-1)$ .
- The off-diagonal elements of the singlet splitting functions matrix have to be multiplied by  $C_F/(2N_f T_f)$  for  $P_{qg}$  and  $2N_f T_f/C_F$  for  $P_{gq}$ , respectively, accounting for the interchange of the initial and final state particles under crossing.

Note that these transformations are automatically accounted for in the case of the leading order physical evolution kernels discussed in the previous paragraph. Keeping this in mind and using the known splitting functions [31, 33, 35] and the continuation rules

$$\ln(1-z) \rightarrow \ln(1-z) - \ln(z) + i\pi, \quad (65)$$

$$\ln(\delta) \rightarrow \ln(\delta) + i\pi, \quad (66)$$

one finds that

$$\bar{P}_{qq}^{(1)S} - P_{qq}^{(1)T} = -2\beta_0 Z_{qq}^{(T)(1)} + Z_{qq}^{(T)(1)} \otimes \bar{P}_{qq}^{(0)} - Z_{qq}^{(T)(1)} \otimes \bar{P}_{qq}^{(0)}, \quad (67)$$

$$\bar{P}_{qg}^{(1)S} - P_{qg}^{(1)T} = -2\beta_0 Z_{qg}^{(T)(1)} + Z_{qg}^{(T)(1)} \otimes (\bar{P}_{gg}^{(0)} - \bar{P}_{qq}^{(0)}) + \bar{P}_{qg}^{(0)} \otimes (Z_{qq}^{(T)(1)} - Z_{gg}^{(T)(1)}), \quad (68)$$

$$\bar{P}_{gq}^{(1)S} - P_{gq}^{(1)T} = -2\beta_0 Z_{gq}^{(T)(1)} + Z_{gq}^{(T)(1)} \otimes (\bar{P}_{qq}^{(0)} - \bar{P}_{gg}^{(0)}) + \bar{P}_{gq}^{(0)} \otimes (Z_{gg}^{(T)(1)} - Z_{qq}^{(T)(1)}), \quad (69)$$

$$\bar{P}_{gg}^{(1)S} - P_{gg}^{(1)T} = -2\beta_0 Z_{gg}^{(T)(1)} + Z_{gg}^{(T)(1)} \otimes \bar{P}_{gg}^{(0)} - Z_{gg}^{(T)(1)} \otimes \bar{P}_{gg}^{(0)}, \quad (70)$$

where the quantities with a bar denote that they are continued from  $z \rightarrow 1/z$  with the appropriate factors in front. These quantities read in explicit form :

$$\begin{aligned} \bar{P}_{qq}^{(n)}(z) &= -z P_{qq}^{(n)}\left(\frac{1}{z}\right) & \bar{P}_{qg}^{(n)}(z) &= \frac{C_F}{2N_f T_f} z P_{qg}^{(n)}\left(\frac{1}{z}\right) \\ \bar{P}_{gq}^{(n)}(z) &= \frac{2N_f T_f}{C_F} z P_{gq}^{(n)}\left(\frac{1}{z}\right) & \bar{P}_{gg}^{(n)}(z) &= -z P_{gg}^{(n)}\left(\frac{1}{z}\right). \end{aligned} \quad (71)$$

The relations given in Eqs. (67–71) remain true for the polarized splitting functions [32, 34] as well. The renormalization factors appearing in the Eqs. (67–70) are given by

$$Z_{ij}^{T(1)} = P_{ji}^{(0)}(\ln(z) + a_{ji}). \quad (72)$$

The constants  $a_{ij}$  are different in the unpolarized and polarized case. For unpolarized scattering they read

$$a_{qq} = a_{gg} = 0, \quad a_{qg} = -\frac{1}{2}, \quad a_{gq} = \frac{1}{2}, \quad (73)$$

whereas in the polarized case

$$a_{ij} = 0 . \quad (74)$$

The logarithms in the renormalization factors originate from the kinematics. In dimensional regularization, when one continues the partonic structure function  $\hat{\mathcal{F}}_{i,k}$  in Eq. (12) from the space-like to the time-like region one obtains an additional factor  $z^\epsilon$  which when multiplied with the pole in  $\epsilon$  yields  $\ln(z)$ . Since the pole is always associated with the splitting functions, one has the function  $P_{ij}^{(0)}$  along with  $\ln(z)$ . The  $z$ -independent constant  $a_{ij}$ , which is also multiplied by the splitting function, results from the polarization average. For deep inelastic scattering one averages the processes with one gluon in the initial state by a factor  $1/(\epsilon + 2)$ . Such an average is not needed for the annihilation process since here the gluon appears in the final state. Notice that the average over the polarization sum does not show up in the polarized structure functions. Therefore in this case the constants  $a_{ij}$  are zero.

The transformation behavior of the non-singlet splitting functions in NLO have been worked out in [33] where also the relations for the NLO non-singlet coefficient functions were presented.

### 4.3 NLO Coefficient Functions

Now, let us study how space-like and time-like coefficient functions are related. The coefficient functions are expected to violate the DLY-relation due to their scheme dependence. Here we first present the relations between the space-like and time-like coefficient functions  $C_{i,k}(z)$  ( $i = 1, L; k = q, g$ ). The leading order transverse coefficient functions are identical. At next-to-leading order, in the  $\overline{\text{MS}}$  scheme [36], the coefficient functions are related by the  $Z$ -factors in Eq. (72) as follows :

$$C_{1,q}^{(T)(1)}(z) + \left\{ z C_{1,q}^{(S)(1)} \left( \frac{1}{z} \right) \right\} = Z_{qq}^{(T)(1)} \quad (75)$$

$$\frac{1}{2} \left[ C_{1,g}^{(T)(1)}(z) - \frac{C_F}{2N_f T_f} \left\{ 2z C_{1,g}^{(S)(1)} \left( \frac{1}{z} \right) \right\} \right] = Z_{qg}^{(T)(1)} . \quad (76)$$

Since the coefficient functions depend on the hard scale of the process, one has to replace the space-like  $q^2$  by the time-like  $q^2$  in addition to Eqs. (65–66). This leads to the following continuation rule

$$\ln \left( \frac{Q^2}{\mu_f^2} \right)_{\text{space-like}} \rightarrow \ln \left( \frac{Q^2}{\mu_f^2} \right)_{\text{time-like}} - i\pi . \quad (77)$$

The  $Z$ -factors get contributions from two sources. The first one is  $z$ -dependent and comes from the phase space integrals. The time-like phase space acquires an extra factor  $z^\epsilon$  which gives a finite contribution when being multiplied with the pole terms  $1/\epsilon$ . The pole term originates from the collinear divergence in  $n$ -dimensional regularization. The second term originates from the polarization average which is again absent in the time-like case. The continuation rules given in Eqs. (65–64, 77) are essential to get the constant  $\zeta(2)$  right when one goes from the space-like to the time-like region. Notice that the space-like coefficient function contains  $-4\zeta(2)\delta(1-z)$  and the time-like one contains  $8\zeta(2)\delta(1-z)$ . The difference which is  $12\zeta(2)$  can be understood to originate from the one-loop vertex correction when one continues from the space-like to time-like region in  $Q^2$ . The same also holds when other regularization methods for the collinear divergences are chosen. It is worth noticing that if one would replace  $\ln(1-z) \rightarrow \ln(1-z) - \ln(z)$  contrary



to the prescription in Eq. (65) one would obtain an additional term  $12\zeta(2)$  on the righthand side of Eq. (75).

The zeroth order longitudinal coefficient functions are identically zero so that the first order contributions are scheme independent. This implies that there are no pole terms in the corresponding partonic structure function  $\hat{\mathcal{F}}_{L,k}$ . Hence, there is no left-over finite piece which could arise from the  $z^\epsilon$ - or  $n$ -dimensional polarization average. We find

$$C_{L,q}^{(T)(1)}(z) - \frac{z}{2} C_{L,q}^{(S)(1)}\left(\frac{1}{z}\right) = 0, \quad (78)$$

$$\frac{1}{2} \left[ C_{L,g}^{(T)(1)}(z) + \frac{C_F}{2N_f T_f} \left\{ z C_{L,g}^{(S)(1)}\left(\frac{1}{z}\right) \right\} \right] = 0. \quad (79)$$

## 4.4 NNLO Coefficient Functions

### 4.4.1 Longitudinal Coefficient Functions

We consider the NNLO correction to the longitudinal coefficient function. We follow the results given in [37, 39, 40] for the space-like and [42, 43] for the time-like case. It turns out that the coefficient functions are related by the  $Z$ -factors through the matrix-valued convolutions

$$C_{L,q}^{(T)(2)}(z) + \left\{ -\frac{z}{2} C_{L,q}^{(S)(2)}\left(\frac{1}{z}\right) \right\} = Z_{qq}^{(T)(1)} \otimes \frac{z}{2} C_{Lq}^{(1)(S)}\left(\frac{1}{z}\right) + Z_{gq}^{(T)(1)} \otimes \frac{C_F}{2N_f T_f} \left\{ -\frac{z}{2} C_{L,g}^{(S)(1)}\left(\frac{1}{z}\right) \right\}, \quad (80)$$

$$\frac{1}{2} \left[ C_{L,g}^{(T)(2)}(z) + \frac{C_F}{2N_f T_f} \left\{ z C_{L,g}^{(S)(2)}\left(\frac{1}{z}\right) \right\} \right] = Z_{gg}^{(T)(1)} \otimes \frac{z}{2} C_{L,q}^{(S)(1)}\left(\frac{1}{z}\right) + Z_{gq}^{(T)(1)} \otimes \frac{C_F}{2N_f T_f} \left\{ -\frac{z}{2} C_{L,g}^{(1)S}\left(\frac{1}{z}\right) \right\}. \quad (81)$$

The right hand side of the above equation contains the convolutions of  $Z$ -factors with the continued NLO longitudinal space-like coefficient functions. We have found this pattern by comparing the scheme transformation which we derived in the last section. The reason for this structure relies on the fact that  $C_{L,i}$  is obtained as the difference between  $C_{2,i}$  and  $C_{1,i}$ . Since the NLO coefficient functions involve various Nielsen-integrals, we used the following identities to simplify the expressions :

$$\text{Li}_2\left(-\frac{1}{z}\right) = -\text{Li}_2(-z) - \frac{1}{2} \ln^2(z) - \zeta(2), \quad (82)$$

$$\text{Li}_2\left(1 - \frac{1}{z}\right) = -\frac{1}{2} \ln^2(z) - \text{Li}_2(1 - z), \quad (83)$$

$$\text{S}_{1,2}\left(1 - \frac{1}{z}\right) = -\frac{1}{6} \ln^3(z) + \text{S}_{1,2}(1 - z), \quad (84)$$

$$\text{Li}_3\left(1 - \frac{1}{z}\right) = \frac{1}{6}\ln^3(z) + S_{1,2}(1-z) - \text{Li}_3(1-z) + \ln(z)\text{Li}_2(1-z), \quad (85)$$

$$\text{Li}_3\left(-\frac{1}{z}\right) = \text{Li}_3(-z) + \frac{1}{6}\ln^3(z) + \zeta(2)\ln(z), \quad (86)$$

$$S_{1,2}\left(-\frac{1}{z}\right) = -S_{1,2}(-z) + \text{Li}_3(-z) - \ln(z)\text{Li}_2(-z) - \frac{1}{6}\ln^3(z) + \zeta(3). \quad (87)$$

If we do not continue  $\ln(1-z)$  and replace  $\ln(1-z) \rightarrow \ln(1-z) - \ln(z)$ , then terms proportional to  $\zeta(2)$  are not compensated between space-like and time-like coefficient functions and hence the relations given in Eqs. (80, 81) are no longer true. Although formally of NNLO, the coefficient functions  $C_{Lq(G)}^{(2)S,T}(z)$  may be combined to physical evolution kernels together with the NLO splitting functions as shown in section 3.1. In Section 4.5. we will show that because of the transformation in Eqs. (80, 81) the physical evolution kernels in sections 3.1 and 3.2 remain DLY-invariant.

#### 4.4.2 Transverse Coefficient Functions

In NNLO physical evolution kernels for the transverse structure and fragmentation function can only be constructed when the space-like and time-like three-loop splitting functions are known. If they become available one can extend Eqs. (47–51) up to second order. Here we consider the relation between the space- and time-like coefficient functions using the transformation relations (64, 65, 66) and (77) for unpolarized and polarized scattering.

The space-like coefficient functions for unpolarized scattering are computed in [39, 40] whereas the time-like ones can be found in [42, 43]. The transverse coefficient functions are related by (see appendix 6.1) :

$$\begin{aligned} C_{1,q}^{(T)(2)}(z) + \left\{ z C_{1,q}^{(S)(2)}\left(\frac{1}{z}\right) \right\} &= \frac{1}{4} \left[ 2(-z)P_{qq}^{(1)S}\left(\frac{1}{z}\right) + 2\beta_0 Z_{qq}^{(T)(1)} + Z_{gq}^{(T)(1)} \otimes \bar{P}_{gq}^{(0)} \right. \\ &\quad \left. - Z_{gq}^{(T)(1)} \otimes \bar{P}_{gq}^{(0)} \right] \ln(z) + \frac{1}{2} Z_{qq}^{(T)(1)} \otimes Z_{qq}^{(T)(1)} \\ &\quad + Z_{qq}^{(T)(1)} \otimes \left( -z C_{1,q}^{(S)(1)}\left(\frac{1}{z}\right) \right) + \frac{1}{2} Z_{gq}^{(T)(1)} \otimes Z_{gq}^{(T)(1)} \\ &\quad + Z_{gq}^{(T)(1)} \otimes \left( \frac{C_F}{2N_f T_f} z C_{1,g}^{(S)(1)}\left(\frac{1}{z}\right) \right) + \frac{1}{8} \bar{P}_{gq}^{(0)} \otimes \bar{P}_{gq}^{(0)} . \\ &\quad + 12C_F^2 \zeta(2) \left( 2 \ln\left(\frac{Q^2}{\mu_f^2}\right) - 3 \right)^2 \delta(1-z) \end{aligned} \quad (88)$$

For the polarized NNLO coefficient functions which were derived in [39, 40] and [42, 43], we find that the form of Eqs. (88) is the same but the term

$$\frac{1}{8} \bar{P}_{gq}^{(0)} \otimes \bar{P}_{gq}^{(0)}$$

does not occur.

Similarly for the gluonic coefficient functions we find

$$\frac{1}{2} \left[ C_{1,g}^{(T)(2)}(z) - \frac{C_F}{2N_f T_f} \left\{ 2z C_{1,g}^{(S)(2)}\left(\frac{1}{z}\right) \right\} \right] = \frac{1}{4} \left[ \frac{C_F}{2N_f T_f} 2z P_{gq}^{(S)(1)}\left(\frac{1}{z}\right) + 2\beta_0 Z_{gq}^{(T)(1)} \right]$$

$$\begin{aligned}
& + Z_{qg}^{(T)(1)} \otimes \bar{P}_{qg}^{(0)} - Z_{qg}^{(T)(1)} \otimes \bar{P}_{qg}^{(0)} + Z_{gg}^{(T)(1)} \otimes \bar{P}_{qg}^{(0)} \\
& - Z_{qg}^{(T)(1)} \otimes \bar{P}_{gg}^{(0)} \left[ \ln(z) + \frac{1}{2} \right] + \frac{1}{2} Z_{qg}^{(T)(1)} \otimes Z_{qg}^{(T)(1)} \\
& + Z_{qg}^{(T)(1)} \otimes \left( -z C_{1,q}^{(S)(1)} \left( \frac{1}{z} \right) \right) + \frac{1}{2} Z_{qg}^{(T)(1)} \otimes Z_{gg}^{(T)(1)} \\
& + Z_{gg}^{(T)(1)} \otimes \left( \frac{C_F}{2N_f T_f} z C_{1,g}^{(S)(1)} \left( \frac{1}{z} \right) \right) \\
& - \frac{1}{8} \beta_0 \bar{P}_{qg}^{(0)} + \frac{1}{16} \bar{P}_{qg}^{(0)} \otimes (\bar{P}_{gg}^{(0)} - \bar{P}_{qg}^{(0)}) . \tag{89}
\end{aligned}$$

For the polarized case, the terms

$$- \frac{1}{8} \beta_0 \bar{P}_{qg}^{(0)} + \frac{1}{16} \bar{P}_{qg}^{(0)} \otimes (\bar{P}_{gg}^{(0)} - \bar{P}_{qg}^{(0)})$$

in Eq. (89) are absent. Since we do not have to average over the initial state gluon polarization in the case of polarized scattering the term  $\ln(z) + 1/2$  multiplying the first bracket in Eq. (89) is replaced by  $\ln(z)$ , cf. also Eq. (72, 74).

## 4.5 NLO Physical Evolution Kernels

After having found the relations between space-like and time-like splitting and coefficient functions, we investigate the DLY-transformation for the physical evolution kernels presented in sections 3.1 and 3.2. In order to do this, we define the difference between the time-like quantities  $K_{IJ}^T$  and the continued space-like quantities  $\bar{K}_{ij}^S$  by

$$\delta K_{IJ} = K_{IJ}^T - \bar{K}_{IJ}^S , \tag{90}$$

where  $\bar{K}_{IJ}^S$  is obtained by transforming  $K_{IJ}^S$  to the time-like region using the continuation rules (65, 66, 64, 77). Application of the DLY-transformations provides us with the following results

$$\begin{aligned}
\delta K_{22}^{N(1)} &= \delta \gamma_{qq}^{N(1)} - \frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}} \delta \gamma_{gq}^{N(1)} + \frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}} \delta C_{2,g}^{N(1)} \bar{\gamma}_{qq}^{N(0)} \\
&\quad - \left[ \frac{\delta C_{L,q}^{N(2)}}{\bar{C}_{L,g}^{N(1)}} + \left( \frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}} \right)^2 \delta C_{2,g}^{N(1)} - \frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}} \frac{\delta C_{L,g}^{N(2)}}{\bar{C}_{L,g}^{N(1)}} \right] \bar{\gamma}_{qq}^{N(0)} \\
&\quad + \bar{\gamma}_{gq}^{N(0)} \delta C_{2,g}^{N(1)} - \frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}} \bar{\gamma}_{gg}^{N(0)} \delta C_{2,g}^{N(1)} + 2\beta_0 \left( \delta C_{2,q}^{N(1)} - \frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}} \delta C_{2,g}^{N(1)} \right) \\
&= \delta \gamma_{qq}^{N(1)} - 2\beta_0 Z_{qg}^{(T)N(1)} - \bar{\gamma}_{qg}^{N(0)} Z_{qg}^{(T)N(1)} + \bar{\gamma}_{qg}^{N(0)} Z_{gq}^{(T)N(1)} \\
&\quad + \frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}} (-\delta \gamma_{gq}^{N(1)} + 2\beta_0 Z_{qg}^{(T)N(1)} - Z_{qg}^{(T)N(1)} \bar{\gamma}_{qq}^{N(0)} + Z_{qg}^{(T)N(1)} \bar{\gamma}_{qg}^{N(0)} \\
&\quad - Z_{gg}^{(T)N(1)} \bar{\gamma}_{qg}^{N(0)} + Z_{qg}^{(T)N(1)} \bar{\gamma}_{gg}^{N(0)}) . \tag{91}
\end{aligned}$$

Substituting the expressions for  $\delta\gamma_{qq}^{N(1)}$  and  $\delta\gamma_{gg}^{N(1)}$  using Eqs. (67, 69), we get

$$\delta K_{22}^{N(1)} = 0 . \quad (92)$$

For the remaining evolution kernels one obtains

$$\delta K_{2L}^{N(1)} = \delta\gamma_{gq}^{N(1)} - 2\beta_0 Z_{qg}^{(T)N(1)} + Z_{qg}^{(T)N(1)}(\bar{\gamma}_{qq}^{N(0)} - \bar{\gamma}_{gg}^{N(0)}) - \bar{\gamma}_{qg}^{N(0)}(Z_{qq}^{(T)N(1)} - Z_{gg}^{(T)N(1)}) , \quad (93)$$

$$\begin{aligned} \delta K_{LL}^{N(1)} &= \delta\gamma_{gg}^{N(1)} - 2\beta_0 \bar{\gamma}_{gg}^{N(0)} + Z_{qg}^{(T)N(1)} \bar{\gamma}_{gq}^{N(0)} - Z_{gg}^{(T)N(1)} \bar{\gamma}_{qg}^{N(0)} \\ &+ \frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}} \left[ \delta\gamma_{gq}^{N(1)} - 2\beta_0 Z_{qg}^{(T)N(1)} + Z_{qg}^{(T)N(1)} \bar{\gamma}_{qq}^{N(0)} - Z_{qg}^{(T)N(1)} \bar{\gamma}_{gg}^{N(0)} - Z_{qg}^{(T)N(1)} \bar{\gamma}_{qg}^{N(0)} \right. \\ &\left. + Z_{gg}^{(T)N(1)} \bar{\gamma}_{qg}^{N(0)} \right] , \end{aligned} \quad (94)$$

$$\begin{aligned} \delta K_{L2}^{N(1)} &= \delta\gamma_{qg}^{N(1)} + \frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}} \delta\gamma_{qq}^{N(1)} - \left( \frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}} \right)^2 \delta\gamma_{gg}^{N(1)} - \frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}} \delta\gamma_{gg}^{N(1)} \\ &- 2\beta_0 Z_{qg}^{(T)N(1)} - Z_{qg}^{(T)N(1)} \bar{\gamma}_{qq}^{N(0)} + Z_{qg}^{(T)N(1)} \bar{\gamma}_{qg}^{N(0)} - Z_{gg}^{(T)N(1)} \bar{\gamma}_{qg}^{N(0)} + Z_{qg}^{(T)N(1)} \bar{\gamma}_{gg}^{N(0)} \\ &+ \frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}} \left[ -2\beta_0 Z_{qg}^{(T)N(1)} - Z_{qg}^{(T)N(1)} \bar{\gamma}_{qq}^{N(0)} + Z_{qg}^{(T)N(1)} \bar{\gamma}_{qg}^{N(0)} \right] \\ &+ \left( \frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}} \right)^2 \left[ 2\beta_0 Z_{qg}^{(T)N(1)} - Z_{qg}^{(T)N(1)} \bar{\gamma}_{qq}^{N(0)} - Z_{gg}^{(T)N(1)} \bar{\gamma}_{qg}^{N(0)} + Z_{qg}^{(T)N(1)} \bar{\gamma}_{gg}^{N(0)} \right. \\ &\left. + Z_{qg}^{(T)N(1)} \bar{\gamma}_{gg}^{N(0)} \right] + \frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}} \left[ 2\beta_0 Z_{gg}^{(T)N(1)} - Z_{qg}^{(T)N(1)} \bar{\gamma}_{qg}^{N(0)} + Z_{qg}^{(T)N(1)} \bar{\gamma}_{qg}^{N(0)} \right] . \end{aligned} \quad (95)$$

The explicit expressions for the differences in the coefficient function are given in appendix 6.1 as well as a series of involved Mellin-convolutions leading to Nielsen-integrals (see appendix 6.2), which are necessary in the explicit calculation.

Using Eqs. (67–70), leads to

$$\delta K_{L2}^{N(1)} = 0 , \quad (96)$$

$$\delta K_{2L}^{N(1)} = 0 , \quad (97)$$

$$\delta K_{LL}^{N(1)} = 0 . \quad (98)$$

The physical evolution kernels  $K_{I,J}$  for the evolution of the structure functions  $F_2$  and  $F_L$  are thus DLY-invariant to next-to-leading order if continued from the space-like to the time-like region.

We turn now to the physical evolution kernels in next-to-leading order where we choose the physical quantities  $F_2$ ,  $\partial F_2 / \partial t$  as a basis. Here only two evolution kernels are contributing, which

change under the DLY-transformation as follows :

$$\begin{aligned}
\delta K_{d2} = & \frac{\beta_0}{2} \left( \delta C_{2q}^{N(1)} - Z_{qq}^{(T)N(1)} \right) \left( \bar{\gamma}_{qq}^{N(0)} + \bar{\gamma}_{gg}^{N(0)} - 2\beta_0 \right) \\
& - \frac{\beta_0}{2\bar{\gamma}_{gq}^{N(0)}} \left( \delta C_{2g}^{N(1)} - Z_{gq}^{(T)N(1)} \right) \left( (\bar{\gamma}_{qq}^{N(0)})^2 - \bar{\gamma}_{qq}^{N(0)} \bar{\gamma}_{gg}^{N(0)} \right. \\
& \left. + 2\bar{\gamma}_{gg}^{N(0)} \bar{\gamma}_{gq}^{N(0)} - 2\beta_0 \bar{\gamma}_{qq}^{N(0)} \right)
\end{aligned} \tag{99}$$

$$\begin{aligned}
\delta K_{dd} = & -2 \frac{\beta_0}{\bar{\gamma}_{gq}^{N(0)}} \left( \delta C_{2g}^{N(1)} - Z_{gq}^{(T)N(1)} \right) \left( \bar{\gamma}_{qq}^{N(0)} - \bar{\gamma}_{gg}^{N(0)} - 2\beta_0 \right) \\
& + 4\beta_0 \left( \delta C_{1q}^{N(1)} - Z_{qq}^{(T)N(1)} \right).
\end{aligned} \tag{100}$$

From Eqs. (75, 76, 78, 79) we can derive that

$$\delta K_{d2} = 0, \tag{101}$$

$$\delta K_{dd} = 0. \tag{102}$$

From these results it is clear that the time-like physical evolution kernels  $K_{ij}^T$  can be directly derived from the space-like physical evolution kernels using the continuations in Eqs. (65, 66, 64, 77) where one has to account for the corresponding changes in the overall color factors. The  $Z^T$ -factors which are needed for the transformation of the splitting and coefficient functions cancel in the expression above. In the future one can extend the investigation performed in this section to physical evolution kernels at the NNLO-level, provided the 3-loop anomalous dimensions are calculated. For the choice of observables ( $F_2, F_L$ ) one also needs the three-loop coefficient functions.

We finally would like to comment on a relation derived by Gribov and Lipatov in [19] for the leading order kernels for a pseudoscalar and a vector field theory.<sup>6</sup> One may write it in the form

$$\overline{K}(x_E, Q^2) = K(x_B, Q^2), \tag{103}$$

where  $\overline{K}$  and  $K$  denote the time- and space-like evolution kernels, respectively, and  $x_B = 1/x_E$ . One verifies, that this relation holds in leading order for the space- and time-like splitting functions of QCD, Eqs. (56–62), without changing the  $\delta$ -function, Eq. (64).

Starting with next-to-leading order, this relation is not preserved. For the physical non-singlet evolution kernels this was shown in [33, 35] and for some singlet combinations in [35]. We find, that also for the physical singlet combinations, Eqs. (42–45, 48–51), this relation is violated as well.

## 5 Conclusions

The old question, whether the scattering cross sections of deep inelastic scattering  $e^- + P \rightarrow e^- + 'X'$  are related to the annihilation cross section  $e^+ + e^- \rightarrow \bar{P} + 'X'$  by a crossing relation changing

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<sup>6</sup> See also [44] for related work.

from  $t$ - to  $s$ -channel was newly discussed. Since in both reactions non-perturbative quantities such as the structure and fragmentation functions contribute the above question cannot be answered by means of perturbation theory for the process as a whole. However, since both the parton densities involved in the space- and time-like process factorize if the virtuality  $Q^2 = |q^2|$  of the four-momentum transfer is large a related question can be asked for the crossing behavior of the respective evolution kernels, which are computable within perturbation theory. In the calculation of both inclusive processes only two types of singularities occur, the collinear singularity and the ultraviolet singularity. These divergences are absorbed into the bare parton densities and the coupling constant, respectively. Two distinct renormalization group equations are implied. They quantify the impact of the factorization and the renormalization scale on the DIS structure functions and fragmentation functions when the perturbation series is truncated up to a given order. However one can construct factorization-scale independent evolution kernels which describe the scheme-invariant evolution of these physical quantities in terms of a kinematic variable given by  $Q^2$ . This scheme invariant evolution is guaranteed up to any finite order in perturbation theory. Notice that in finite order this method does not remove the dependence of the physical quantities on the renormalization scheme of the strong coupling constant or its scale  $\mu_r$ .

The first example of the application of the physical evolution kernels is the coupled structure functions  $F_2(x, Q^2)$  and  $F_L(x, Q^2)$  associated with the corresponding fragmentation functions in  $e^+e^-$ -annihilation. A second example is given by  $F_2(x, Q^2)$  and  $\partial F_2(x, Q^2)/\partial \ln(Q^2)$ . Contrary to the splitting functions (anomalous dimensions) and coefficient functions the evolution kernels of the examples above are factorization scheme independent. For that purpose transformation relations have been derived for the splitting functions up to NLO and the coefficient functions up to NNLO. We have also shown that these kernels are invariant under the Drell-Levy-Yan-transformation up to next-to-leading order. On the other hand the Gribov-Lipatov relation, which is valid in leading order, is already violated at next-to-leading order. It remains to be seen how the physical evolution kernels behave under the DLY crossing relation at NNLO, which presumes the knowledge of the yet unknown three-loop splitting functions (space- and time-like) as well the three-loop longitudinal coefficient functions in the first example above.

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## 6 Appendix

### 6.1 Coefficient Functions

In this appendix we list the difference of the space- and time-like coefficient functions in the  $\overline{\text{MS}}$  scheme, which are used in section 4 to study the validity of the DLY-relation. Here the expressions also contain the logarithms

$$L_{\mu_f} = \ln(Q^2/\mu_f^2) \quad (104)$$

which arise when the factorization scale  $\mu_f^2$  is chosen to be different from  $Q^2$ .

The difference between the longitudinal non-singlet coefficient functions corresponding to the processes  $\gamma^* + q \rightarrow q + g + g$  and  $\gamma^* \rightarrow 'q' + q + g + g$  respectively are given by

$$\delta C_{L,q}^{(2)NS} = C_F^2 \left[ 4(2z - \ln(z)) \ln(z) - 16\text{Li}_2(1-z) + 8 - 8z \right] \quad (105)$$

where ‘ $q'$ ’ denotes the quark in the final state which undergoes fragmentation into a hadron  $P$  (see Eq. (8)). The difference between the longitudinal purely singlet coefficient functions corresponding to the processes  $\gamma^* + q \rightarrow q + q + \bar{q}$  and  $\gamma^* \rightarrow 'q' + q + q + \bar{q}$  respectively are given by

$$\begin{aligned} \delta C_{L,q}^{(2)PS} = & N_f T_f C_F \left[ -8 \left( 6 + 4z - \frac{4}{3} z^2 \right) \ln(z) + 16 \ln^2(z) - 16 \right. \\ & \left. - \frac{304}{9z} + 64z - \frac{128}{9} z^2 \right] \end{aligned} \quad (106)$$

The same is done for the longitudinal gluonic coefficient functions corresponding to the processes  $\gamma^* + g \rightarrow g + q + \bar{q}$  and  $\gamma^* \rightarrow 'g' + g + q + \bar{q}$ , respectively. The difference in the coefficient function is given by

$$\begin{aligned} \delta C_{L,g}^{(2)} = & C_F^2 \left[ 8 \left( 1 + \frac{2}{z} - z \right) \ln(z) + 8 \ln^2(z) - 28 + \frac{24}{z} + 4z \right] \\ & + C_A C_F \left[ 16 \left( 4 - \frac{2}{z} + z - \frac{1}{3} z^2 \right) \ln(z) - 16 \left( 1 + \frac{1}{z} \right) \ln^2(z) \right. \\ & \left. + 32 \left( 1 - \frac{1}{z} \right) \text{Li}_2(1-z) - 8 + \frac{248}{9z} - 24z + \frac{40}{9} z^2 \right]. \end{aligned} \quad (107)$$

Notice that for the computation of the coefficients functions above and the ones following hereafter one also needs the virtual contributions to the zeroth and first order partonic processes. The differences between the transverse coefficient functions emerge from the same processes as mentioned above Eqs. (105, 106, 107). In the non-singlet case we have

$$\begin{aligned} \delta C_{1,q}^{(2)NS} = & \left( C_F^2 - \frac{1}{2} C_A C_F \right) \left[ 8 \frac{\ln(z)}{1+z} \left( -2\zeta(2) - 4 \ln(z) \ln(1+z) + \ln^2(z) \right) \right. \\ & + 4 \left( 2(1-z)\zeta(2) + 4(1-z) + 4(1-z) \ln(z) \ln(1+z) \right. \\ & \left. \left. + 2(1+z) \ln(z) - (1-z) \ln^2(z) \right) \ln(z) - 16 \text{Li}_2(-z) \ln(z) \right. \\ & \left. \times \left( \frac{2}{1+z} - 1 + z \right) \right] \\ & + N_f T_f C_F \left[ \frac{8}{9} \left( -\frac{10}{1-z} - 1 + 11z \right) \ln(z) \right] \\ & + C_A C_F \left[ 4 \frac{\ln(z)}{1-z} \left( \frac{67}{9} + \ln^2(z) - 2\zeta(2) \right) + 2 \left( \frac{53}{9} - \frac{187}{9} z \right. \right. \\ & \left. \left. + 2(1+z) \ln(z) - (1+z) \ln^2(z) \right) \ln(z) + 4\zeta(2)(1+z) \ln(z) \right] \\ & + C_F^2 \left[ 4 \frac{\ln(z)}{1-z} \left( (8L_{\mu_f} - 6 + 4 \ln(1-z)) \ln(1-z) + 6L_{\mu_f} - 18 \right) \right. \end{aligned}$$

$$\begin{aligned}
& + \left( -4L_{\mu_f} + 6 - \frac{16}{3} \ln(z) \right) \ln(z) \Bigg) + 4 \left( -4(1+z)L_{\mu_f} + 1 + 5z \right. \\
& \left. - 2(1+z) \ln(1-z) \right) \ln(z) \ln(1-z) + 2 \left( -2(5+z)L_{\mu_f} + 14 + 40z \right. \\
& \left. + 2(1+z) \ln(z) \ln(1-z) + 6(1+z)L_{\mu_f} \ln(z) - 8(2+z) \ln(z) \right. \\
& \left. + 7(1+z) \ln^2(z) \right) \ln(z) + 2\text{Li}_2(1-z) \left( 4 \left( -\frac{12}{1-z} + 7 + 7z \right) \ln(z) \right. \\
& \left. - 10 + 22z \right) + 8 \left( -\frac{24}{1-z} + 13 + 13z \right) \text{S}_{1,2}(1-z) + 36(1-z) \\
& \left. + 32\zeta(2) \left( \frac{2}{1-z} - 1 - z \right) \ln(z) + 12\zeta(2)(9 - 12L_{\mu_f} + 4L_{\mu_f}^2) \delta(1-z) \right] . \quad (108)
\end{aligned}$$

For the purely singlet difference we obtain

$$\begin{aligned}
\delta C_{1,q}^{(2)PS} &= N_f T_f C_F \left[ \left( -8 \left( 4 + 6z + \frac{8}{3} z^2 \right) (L_{\mu_f} + \ln(1-z)) - 16 \left( 1 - \frac{2}{3z} + 2z \right) \right. \right. \\
&\quad \times \ln(z) - 160 - \frac{160}{9z} - 112z - \frac{368}{9} z^2 + 4(1+z) \left( 4 \ln(1-z) + 4L_{\mu_f} \right. \\
&\quad \left. \left. + \frac{10}{3} \ln(z) \right) \ln(z) \right) \ln(z) - 8 \left( 1 + \frac{38}{9z} - z - \frac{38}{9} z^2 \right) \left( L_{\mu_f} + \ln(1-z) \right) \\
&\quad + 16 \left( 2(1+z) \ln(z) - 2 - 3z - \frac{4}{3} z^2 \right) \text{Li}_2(1-z) + 32(1+z) \text{S}_{1,2}(1-z) \\
&\quad \left. \left. - \frac{1168}{9} - \frac{224}{27z} + \frac{640}{9} z + \frac{1808}{27} z^2 \right] \right] . \quad (109)
\end{aligned}$$

The difference between the gluonic coefficient functions equals

$$\begin{aligned}
\delta C_{1,g}^{(2)} &= 2C_A C_F \left[ \left( 188 + \frac{704}{9z} + 66z + \frac{184}{9} z^2 + \left( 48 - \frac{24}{z} + 16z + \frac{32}{3} z^2 \right) L_{\mu_f} \right. \right. \\
&\quad + \left( -4 - \frac{100}{3z} + 10z \right) \ln(z) + \left( 40 - \frac{16}{z} + 12z + \frac{32}{3} z^2 \right) \ln(1-z) \\
&\quad + 4 \left( 2 + \frac{2}{z} + z \right) \ln(1+z) - 16 \left( 1 + \frac{1}{z} + z \right) \ln(z) L_{\mu_f} \\
&\quad + 8 \left( 2 + \frac{2}{z} + z \right) \ln(z) \ln(1+z) - 16 \left( 1 + \frac{1}{z} + z \right) \ln(z) \ln(1-z) \\
&\quad - \left( \frac{40}{3} + \frac{64}{3z} + \frac{52}{3} z \right) \ln^2(z) + 4 \left( 2 - \frac{2}{z} - z \right) \ln^2(1-z) \Bigg) \ln(z) \\
&\quad \left. + \left( -32 + \frac{356}{9z} + 4z - \frac{104}{9} z^2 \right) \ln(1-z) + 2 \left( 2 - \frac{2}{z} - z \right) \ln^2(1-z) \right]
\end{aligned}$$



$$\begin{aligned}
& + \left( -24 + \frac{284}{9z} + 4z - \frac{104}{9}z^2 \right) L_{\mu_f} + \left( 32 \left( 1 - \frac{3}{z} - 2z \right) \ln(z) \right. \\
& + 16 \left( 2 - \frac{2}{z} - z \right) \left( \ln(1-z) + L_{\mu_f} \right) + 16 + \frac{8}{z} + 16z + \frac{32}{3}z^2 \Big) \text{Li}_2(1-z) \\
& + 4 \left( 2 + \frac{2}{z} + z \right) \left( 1 + 2\ln(z) \right) \text{Li}_2(-z) + 16 \left( -2 + \frac{2}{z} + z \right) \text{Li}_3(1-z) \\
& + 8 \left( 8 - \frac{16}{z} - 10z \right) \text{S}_{1,2}(1-z) - \frac{418}{9} + \frac{4168}{27z} - \frac{668}{9}z - \frac{856}{27}z^2 \\
& + 8\zeta(2) \left( -3 + \frac{4}{z} + 2z \right) (1 + 2\ln(z)) \Big] \\
& + 2C_F^2 \left[ \left( 42 + \frac{8}{z} - 29z - (14 - 11z) \ln(z) + 16 \left( 3 - \frac{3}{z} - z \right) \ln(1-z) \right. \right. \\
& + 4(2-z) \ln(z) L_{\mu_f} - 4 \left( 2 - \frac{4}{z} - z \right) \ln(z) \ln(1-z) \\
& - 16 \left( 2 - \frac{2}{z} - z \right) \ln(1-z) L_{\mu_f} + \frac{10}{3}(2-z) \ln^2(z) \\
& - 12 \left( 2 - \frac{2}{z} - z \right) \ln^2(1-z) \Big) \ln(z) + \left( 64 - \frac{52}{z} - 18z \right. \\
& - 8 \left( 2 - \frac{2}{z} - z \right) L_{\mu_f} - 6 \left( 2 - \frac{2}{z} - z \right) \ln(1-z) \Big) \ln(1-z) \\
& + 2 \left( 16 - \frac{10}{z} - 3z \right) L_{\mu_f} + 8 \left( (2-z) \ln(z) - 2 \left( 2 - \frac{2}{z} - z \right) \left( \ln(1-z) \right. \right. \\
& + L_{\mu_f} \Big) + 6 - \frac{6}{z} - 3z \Big) \text{Li}_2(1-z) + 16 \left( 2 - \frac{2}{z} - z \right) \text{Li}_3(1-z) \\
& \left. + 8 \left( 10 - \frac{8}{z} - 5z \right) \text{S}_{1,2}(1-z) - 169 + \frac{106}{z} + 50z \right] . \tag{110}
\end{aligned}$$

We have computed the same differences between the coefficient functions corresponding to the structure function  $g_1(x, Q^2)$  which describes polarized scattering. The analogues of Eqs. (108, 109, 110) are given by

$$\begin{aligned}
\delta\Delta C_{1,q}^{(2)NS} &= \left( C_F^2 - \frac{1}{2}C_A C_F \right) \left[ 8 \frac{\ln(z)}{1+z} \left( -2\zeta(2) - 4\ln(z) \ln(1+z) + \ln^2(z) \right) \right. \\
&+ 4 \left( 2(1-z)\zeta(2) + 4(1-z) + 4(1-z) \ln(z) \ln(1+z) \right. \\
&\left. \left. + 2(1+z) \ln(z) - (1-z) \ln^2(z) \right) \ln(z) - 16\text{Li}_2(-z) \ln(z) \right]
\end{aligned}$$

$$\begin{aligned}
& \times \left( \frac{2}{1+z} - 1 + z \right) \Big] \\
& + N_f T_f C_F \left[ \frac{8}{9} \left( -\frac{10}{1-z} - 1 + 11z \right) \ln(z) \right] \\
& + C_A C_F \left[ 4 \frac{\ln(z)}{1-z} \left( \frac{67}{9} + \ln^2(z) - 2\zeta(2) \right) + 2 \left( \frac{53}{9} - \frac{187}{9} z \right. \right. \\
& \left. \left. + 2(1+z) \ln(z) - (1+z) \ln^2(z) \right) \ln(z) + 4\zeta(2)(1+z) \ln(z) \right] \\
& + C_F^2 \left[ 4 \frac{\ln(z)}{1-z} \left( (8L_{\mu_f} - 6 + 4 \ln(1-z)) \ln(1-z) + 6L_{\mu_f} - 18 \right. \right. \\
& \left. \left. + \left( -4L_{\mu_f} + 6 - \frac{16}{3} \ln(z) \right) \ln(z) \right) + 4 \left( -4(1+z)L_{\mu_f} + 1 + 5z \right. \right. \\
& \left. \left. - 2(1+z) \ln(1-z) \right) \ln(z) \ln(1-z) + 2 \left( -2(5+z)L_{\mu_f} + 18 + 36z \right. \right. \\
& \left. \left. + 2(1+z) \ln(z) \ln(1-z) + 6(1+z)L_{\mu_f} \ln(z) - 2(7+5z) \ln(z) \right. \right. \\
& \left. \left. + 7(1+z) \ln^2(z) \right) \ln(z) + 2\text{Li}_2(1-z) \left( 4 \left( -\frac{12}{1-z} + 7 + 7z \right) \ln(z) \right. \right. \\
& \left. \left. - 2 + 14z \right) + 8 \left( -\frac{24}{1-z} + 13 + 13z \right) \text{S}_{1,2}(1-z) + 36(1-z) \right. \\
& \left. + 32\zeta(2) \left( \frac{2}{1-z} - 1 - z \right) \ln(z) + 12\zeta(2)(9 - 12L_{\mu_f} + 4L_{\mu_f}^2) \delta(1-z) \right] \quad (111)
\end{aligned}$$

$$\begin{aligned}
\delta \Delta C_{1,q}^{(2)PS} &= N_f T_f C_F \left[ \left( 16(1-4z) \left( L_{\mu_f} + \ln(1-z) \right) - 184 + 24z - 16(2+3z) \right. \right. \\
& \times \ln(z) + 4(1+z) \left( 4L_{\mu_f} + 4 \ln(1-z) + \frac{10}{3} \ln(z) \right) \ln(z) \Big) \ln(z) \\
& - 48(1-z) \left( L_{\mu_f} + \ln(1-z) \right) + 16 \left( 2(1+z) \ln(z) + 1 - 4z \right) \text{Li}_2(1-z) \\
& \left. + 32(1+z) \text{S}_{1,2}(1-z) - 112 + 112z \right] \quad (112)
\end{aligned}$$

$$\begin{aligned}
\delta \Delta C_{1,g}^{(2)} &= 2C_A C_F \left[ \left( 212 + 48z - 32(1-2z)L_{\mu_f} + 4(8+5z) \ln(z) - 16(1-3z) \right. \right. \\
& \times \ln(1-z) - 8(4+z) \ln(z)L_{\mu_f} + 8(2+z) \ln(z) \ln(1+z)
\end{aligned}$$

$$\begin{aligned}
& -8(4+z)\ln(z)\ln(1-z) - \frac{4}{3}(26+5z)\ln^2(z) - 4(2-z) \\
& \times \ln^2(1-z) \Big) \ln(z) + 32(1-z) \Big( \ln(1-z) + L_{\mu_f} \Big) \\
& + 16 \Big( 2+z - (2-z) \Big( L_{\mu_f} + \ln(1-z) \Big) - (8-z)\ln(z) \Big) \text{Li}_2(1-z) \\
& + 8(2+z)\ln(z)\text{Li}_2(-z) + 16(2-z)\text{Li}_3(1-z) - 32(5-z)\text{S}_{1,2}(1-z) \\
& + 224 - 224z + 8\zeta(2)(8-3z)\ln(z) \Big] \\
& + 2C_F^2 \Big[ \Big( 22 + 40z + 4(2-z)L_{\mu_f} - 4(14-9z)\ln(1-z) + 4(8-5z)\ln(z) \\
& + 16(2-z)\ln(1-z)L_{\mu_f} - 4(2-z)\ln(z)L_{\mu_f} \\
& + 4(2-z)\ln(z)\ln(1-z) - \frac{10}{3}(2-z)\ln^2(z) \\
& + 12(2-z)\ln^2(1-z) \Big) \ln(z) - 8(1-z) \Big( \ln(1-z) + L_{\mu_f} \Big) \\
& + 4 \Big( -10 + 5z + 4(2-z) \Big( \ln(1-z) + L_{\mu_f} \Big) - 2(2-z)\ln(z) \Big) \\
& \times \text{Li}_2(1-z) - 16(2-z)\text{Li}_3(1-z) - 40(2-z)\text{S}_{1,2}(1-z) + 96 - 96z \Big] . \quad (113)
\end{aligned}$$

## 6.2 Convolutions

Here we list the convolutions of a series of functions, which are needed for the investigation of the DLY-relation in section 4. Using the definition in Eq. (13) we obtain

$$\begin{aligned}
\frac{\ln(z)}{(1-z)} \otimes \frac{\ln(z)}{(1-z)} &= \frac{1}{(1-z)} \Big[ -4 \text{S}_{1,2}(1-z) - 2\ln(z)\text{Li}_2(1-z) \\
&\quad - \frac{1}{6}\ln^3(z) \Big] \quad (114)
\end{aligned}$$

$$\frac{\ln(z)}{(1-z)} \otimes \frac{\ln(z)}{z} = -\frac{1}{z} [2 \text{S}_{1,2}(1-z) + \ln(z)\text{Li}_2(1-z)] \quad (115)$$

$$\begin{aligned}
\frac{\ln(z)}{(1-z)} \otimes z^2 \ln(z) &= \frac{1}{12} \Big[ -3 - 24z + 27z^2 - 24 \text{S}_{1,2}(1-z)z^2 \\
&\quad - 3(1+4z+5z^2)\ln(z) - 2z^2\ln^3(z) \\
&\quad - 12z^2\ln(z)\text{Li}_2(1-z) \Big] \quad (116)
\end{aligned}$$

$$\begin{aligned}\frac{\ln(z)}{(1-z)} \otimes z \ln(z) &= -2 + 2z - 2z S_{1,2}(1-z) - \ln(z) - z \ln(z) \\ &\quad - \frac{z}{6} \ln^3(z) - z \ln(z) \text{Li}_2(1-z)\end{aligned}\quad (117)$$

$$\frac{\ln(z)}{(1-z)} \otimes \ln(z) = -2 S_{1,2}(1-z) - \frac{1}{6} \ln^3(z) - \ln(z) \text{Li}_2(1-z) \quad (118)$$

$$\frac{\ln(z)}{(1-z)} \otimes \frac{\ln(1-z)}{z} = \frac{1}{z} [-S_{1,2}(1-z) - \ln(1-z) \text{Li}_2(1-z) + \text{Li}_3(1-z)] \quad (119)$$

$$\begin{aligned}\frac{\ln(z)}{(1-z)} \otimes z^2 \ln(1-z) &= \frac{1}{4} \left\{ -4 - 5z + 9z^2 - 8 S_{1,2}(1-z) z^2 + \ln(1-z) \right. \\ &\quad + 4z \ln(1-z) - 5z^2 \ln(1-z) - 3 \ln(z) - 4z \ln(z) \\ &\quad + 2 \ln(1-z) \ln(z) + 4z \ln(1-z) \ln(z) \\ &\quad - 2z^2 \ln(1-z) \ln^2(z) + \left[ 2 + 4z - 4z^2 \ln(1-z) \right. \\ &\quad \left. \left. - 4z^2 \ln(z) \right] \text{Li}_2(1-z) + 4z^2 \text{Li}_3(1-z) \right\}\end{aligned}\quad (120)$$

$$\begin{aligned}\frac{\ln(z)}{(1-z)} \otimes z \ln(1-z) &= -2 + 2z - 2z S_{1,2}(1-z) + \ln(1-z) - z \ln(1-z) \\ &\quad - \ln(z) + \ln(1-z) \ln(z) - \frac{z}{2} \ln(1-z) \ln^2(z) \\ &\quad - \left[ -1 + z \ln(1-z) + z \ln(z) \right] \text{Li}_2(1-z) \\ &\quad + z \text{Li}_3(1-z)\end{aligned}\quad (121)$$

$$\begin{aligned}\frac{\ln(z)}{(1-z)} \otimes \ln(1-z) &= -2 S_{1,2}(1-z) - \frac{1}{2} \ln(1-z) \ln^2(z) \\ &\quad - [\ln(1-z) + \ln(z)] \text{Li}_2(1-z) + \text{Li}_3(1-z)\end{aligned}\quad (122)$$

$$\begin{aligned}\frac{\ln(z)}{(1-z)} \otimes \left( \frac{\ln(1-z)}{1-z} \right)_+ &= \frac{1}{(1-z)} \left[ \frac{1}{2} \ln(z) \ln^2(1-z) - 2 S_{1,2}(1-z) \right. \\ &\quad \left. - \ln(z) \text{Li}_2(1-z) - \frac{1}{2} \ln^2(z) \ln(1-z) \right]\end{aligned}\quad (123)$$

$$\frac{\ln(z)}{(1-z)} \otimes \frac{1}{(1-z)_+} = \frac{1}{(1-z)} \left[ \ln(z) \ln(1-z) - \frac{1}{2} \ln^2(z) \right] \quad (124)$$

$$\begin{aligned}\frac{\ln(z)}{z} \otimes \frac{\ln(1-z)}{z} &= \frac{1}{2z} \left[ -2 S_{1,2}(1-z) - \ln(1-z) \ln^2(z) \right. \\ &\quad \left. - 2 \ln(z) \text{Li}_2(1-z) \right]\end{aligned}\quad (125)$$

$$\begin{aligned}\frac{\ln(z)}{z} \otimes \left( \frac{\ln(1-z)}{1-z} \right)_+ &= \frac{1}{z} \left[ \frac{1}{2} \ln^2(1-z) \ln(z) + \ln(1-z) \text{Li}_2(1-z) \right. \\ &\quad \left. - \text{Li}_3(1-z) \right]\end{aligned}\quad (126)$$

$$\frac{\ln(z)}{z} \otimes \frac{1}{(1-z)_+} = \frac{1}{z} [-\text{Li}_2(z) + \zeta(2)] \quad (127)$$

$$\begin{aligned}
z^2 \ln(z) \otimes \left( \frac{\ln(1-z)}{1-z} \right)_+ &= z^2 \left[ -\frac{3}{4} - S_{1,2}(1-z) + \frac{3}{4z} + \frac{5}{4} \ln(1-z) \right. \\
&\quad - \frac{1}{4z^2} \ln(1-z) - \frac{1}{z} \ln(1-z) - \frac{3}{4} \ln(z) \\
&\quad - \frac{3}{2} \ln(1-z) \ln(z) + \frac{1}{2} \ln^2(1-z) \ln(z) \\
&\quad + \frac{3}{4} \ln^2(z) - \frac{1}{2} \ln(1-z) \ln^2(z) - \frac{3}{2} \text{Li}_2(1-z) \\
&\quad + \ln(1-z) \text{Li}_2(1-z) - \ln(z) \text{Li}_2(1-z) \\
&\quad \left. - \text{Li}_3(1-z) \right] \tag{128}
\end{aligned}$$

$$\begin{aligned}
z^2 \ln(z) \otimes z^2 \ln(1-z) &= -\frac{z^2}{2} \left( 2 S_{1,2}(1-z) + \ln(1-z) \ln^2(z) \right. \\
&\quad \left. + 2 \ln(z) \text{Li}_2(1-z) \right) \tag{129}
\end{aligned}$$

$$\begin{aligned}
z^2 \ln(z) \otimes \frac{1}{(1-z)_+} &= -\frac{1}{4} - z + \frac{5}{4} z^2 - \frac{3}{2} z^2 \ln(z) \\
&\quad - \frac{z^2}{2} \ln^2(z) + z^2 \ln(z) \ln(1-z) + z^2 \text{Li}_2(1-z) \tag{130}
\end{aligned}$$

$$\begin{aligned}
z \ln(z) \otimes \left( \frac{\ln(1-z)}{1-z} \right)_+ &= z \left[ -S_{1,2}(1-z) + \ln(1-z) - \frac{1}{z} \ln(1-z) - \ln(z) \right. \\
&\quad + \frac{1}{2} \ln^2(1-z) \ln(z) - \ln(z) \ln(1-z) + \frac{1}{2} \ln^2(z) \\
&\quad - \frac{1}{2} \ln(1-z) \ln^2(z) - \text{Li}_2(1-z) + \ln(1-z) \\
&\quad \left. \times \text{Li}_2(1-z) - \ln(z) \text{Li}_2(1-z) - \text{Li}_3(1-z) \right] \tag{131}
\end{aligned}$$

$$\begin{aligned}
z \ln(z) \otimes \frac{1}{(1-z)_+} &= -z \ln(z) - 1 + z - \frac{z}{2} \ln^2(z) + z \text{Li}_2(1-z) \\
&\quad + z \ln(z) \ln(1-z) \tag{132}
\end{aligned}$$

$$\begin{aligned}
z \ln(z) \otimes z \ln(1-z) &= -z S_{1,2}(1-z) - \frac{z}{2} \ln^2(z) \ln(1-z) \\
&\quad - z \ln(z) \text{Li}_2(1-z) \tag{133}
\end{aligned}$$

$$\begin{aligned}
\ln(z) \otimes \left( \frac{\ln(1-z)}{1-z} \right)_+ &= -S_{1,2}(1-z) + \frac{1}{2} \ln^2(1-z) \ln(z) \\
&\quad - \frac{1}{2} \ln(1-z) \ln^2(z) + \ln(1-z) \text{Li}_2(1-z) \\
&\quad - \ln(z) \text{Li}_2(1-z) - \text{Li}_3(1-z) \tag{134}
\end{aligned}$$

$$\begin{aligned}
\ln(z) \otimes \ln(1-z) &= -S_{1,2}(1-z) - \frac{1}{2} \ln^2(z) \ln(1-z) \\
&\quad - \ln(z) \text{Li}_2(1-z) \tag{135}
\end{aligned}$$

$$\begin{aligned}
\ln(z) \otimes \frac{1}{(1-z)_+} &= -\frac{1}{2} \ln^2(z) + \text{Li}_2(1-z) + \ln(z) \ln(1-z) \tag{136}
\end{aligned}$$

$$\frac{1}{(1-z)_+} \otimes \frac{1}{(1-z)_+} = 2 \frac{\ln(1-z)}{(1-z)} - \frac{\ln(z)}{(1-z)} - \delta(1-z)\zeta(2) \quad (137)$$

$$\frac{1}{z} \otimes \left( \frac{\ln(1-z)}{1-z} \right)_+ = \frac{1}{2z} \ln^2(1-z) \quad (138)$$

$$\frac{1}{z} \otimes \frac{1}{(1-z)_+} = \frac{1}{z} \ln(1-z) \quad (139)$$

$$\begin{aligned} z^2 \otimes \left( \frac{\ln(1-z)}{1-z} \right)_+ &= \left[ \frac{1}{2} + z - \frac{3}{2}z^2 \right] \ln(1-z) - \frac{1}{2}z(1-z) \\ &\quad + \frac{3}{2}z^2 \ln^2(z) + \frac{1}{2}z^2 \ln^2(1-z) - z^2 \ln(z) \ln(1-z) \\ &\quad - z^2 \text{Li}_2(1-z) \end{aligned} \quad (140)$$

$$z^2 \otimes \frac{1}{(1-z)_+} = \frac{1}{2} + z - \frac{3}{2}z^2 - z^2 \ln(z) + z^2 \ln(1-z) \quad (141)$$

$$\begin{aligned} z \otimes \left( \frac{\ln(1-z)}{1-z} \right)_+ &= (1-z) \ln(1-z) + z \ln(z) - z \ln(z) \ln(1-z) \\ &\quad - z \text{Li}_2(1-z) + \frac{1}{2}z \ln^2(1-z) \end{aligned} \quad (142)$$

$$z \otimes \frac{1}{(1-z)_+} = 1 - z + z \ln(1-z) - z \ln(z) \quad (143)$$

$$1 \otimes \left( \frac{\ln(1-z)}{1-z} \right)_+ = -\text{Li}_2(1-z) - \ln(z) \ln(1-z) + \frac{1}{2} \ln^2(1-z) \quad (144)$$

$$1 \otimes \frac{1}{(1-z)_+} = \ln(1-z) - \ln(z) \quad (145)$$

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